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A Riemannian manifold with skew-circulant structures and an associated locally conformal Kähler manifold

Iva Dokuzova[1][2] **and Dimitar Razpopov**[3]

Abstract. A 4-dimensional Riemannian manifold M, equipped with an additional tensor structure S, whose fourth power is minus identity, is considered. The structure S has a skew-circulant matrix with respect to some basis and S acts as an isometry with respect to the metric g. A fundamental tensor is defined on such a manifold (M, g, S) by g and by the covariant derivative of S. This tensor satisfies a characteristic identity which is invariant to the usual conformal transformation. Some curvature properties of (M, g, S) are obtained. A Lie group as a manifold of the considered type is constructed. A Hermitian manifold associated with (M, g, S) is also considered. It turns out that it is a locally conformal Kähler manifold.

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1. Introduction

The classification of almost Hermitian manifolds with respect to the covariant derivative of the almost complex structure J is made by Gray and Hervella in [6]. The Hermitian manifolds form a class of manifolds with an integrable almost complex structure J. Their subclass consists of the so-called locally conformal Kähler manifolds, determined by a special property of the covariant derivative of J. The class of Kähler manifolds is common to all classes in this classification and its manifolds have the richest geometry. The Kähler manifolds are extensively studied by many geometers. There is also a great interest in the study of locally conformal Kähler manifolds, since their structure J satisfies similar but weaker conditions than those of Kähler manifolds. Some of the recent investigations of locally conformal Kähler manifolds are made in [1, 3, 7, 9, 11, 12].

Problems in differential geometry of a 4-dimensional Riemannian manifold M with a tensor structure S of type $(1, 1)$, which satisfies $S^4 = -\mathrm{id}$, are considered in [5]. The matrix of S in some basis is skew-circulant. Moreover, S is compatible with the metric g, so that an isometry is induced in any tangent

[1]Department of Algebra and Geometry, University of Plovdiv Paisii Hilendarski, 24 Tzar Asen, 4000 Plovdiv, Bulgaria, e-mail: dokuzova@uni-plovdiv.bg
[2]Corresponding author
[3]Department of Mathematics and Informatics, Agricultural University of Plovdiv, 12 Mendeleev Blvd., 4000 Plovdiv, Bulgaria, e-mail: razpopov@au-plovdiv.bg

space of M. A Hermitian manifold (M, g, J), where $J = S^2$, is associated with such a manifold (M, g, S).

In the present work, we continue the study of (M, g, S) and (M, g, J). In Section 2, we recall some necessary facts about these manifolds. In Section 3, we compute the components of the fundamental tensor F on (M, g, S) determined by the metric g and by the covariant derivative of S. We obtain an important characteristic identity for F. We establish that the image of the fundamental tensor with respect to the usual conformal transformation satisfies the same identity. In Section 4, we find some curvature properties of (M, g, S). In Section 5, we establish that the associated manifold (M, g, J) belongs to the class of locally conformal Kähler manifolds. In Section 6, we construct a Lie group with a Lie algebra of a special class as a manifold with the structure (g, S).

2. Preliminaries

We consider a 4-dimensional Riemannian manifold M equipped with a tensor S of type $(1, 1)$. The structure S has a skew-circulant matrix, with respect to some basis, given by

$$(S_j^k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Then S has the property

$$S^4 = -\mathrm{id}.$$

The metric g and the structure S satisfy

$$g(Sx, Sy) = g(x, y), \quad x, y \in \mathfrak{X}(M). \tag{2.2}$$

The above condition and (2.1) imply that the matrix of g has the form:

$$(g_{ij}) = \begin{pmatrix} A & B & 0 & -B \\ B & A & B & 0 \\ 0 & B & A & B \\ -B & 0 & B & A \end{pmatrix}. \tag{2.3}$$

Here A and B are smooth functions of an arbitrary point $p(x^1, x^2, x^3, x^4)$ on M. It is supposed that $A > \sqrt{2}B > 0$ in order g to be positive definite. The manifold (M, g, S) is introduced in [5].

Anywhere in this work, x, y, z, u will stand for arbitrary elements of the algebra of the smooth vector fields $\mathfrak{X}(M)$ or vectors in the tangent space T_pM. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3, 4\}$.

In [5], it is noted that the manifold (M, g, J), where $J = S^2$, is a Hermitian manifold with an almost complex structure J. For such manifolds Gray-Hervella's classification is valid ([6]). This classification is made with respect to the covariant derivative of the Kähler form $J(x,y)=g(x,Jy)$. The almost Hermitian manifolds with an integrable structure J are called Hermitian manifolds. Their subclass of locally conformal Kähler manifolds is determined by the property

$$\begin{aligned} g\big((\nabla_x J)y,z\big) =&\frac{1}{2}\big\{(g(x,y)\omega(z)-g(x,z)\omega(y)+g(x,Jy)\omega(Jz) \\ &-g(x,Jz)\omega(Jy)\big\},\quad \omega(x)=g^{ij}g\big((\nabla_{e_i}J)e_j,x\big). \end{aligned} \tag{2.4}$$

Here ∇ is the Levi-Civita connection of g, and g^{ij} are the components of the inverse matrix of (g_{ij}) with respect to the basis $\{e_i\}$ of T_pM.

It is known that the class of 4-dimensional locally conformal Kähler manifolds is non-trivial. Every Kähler manifold belongs to the class of locally conformal Kähler manifolds ([6]).

Now we consider an associated metric $\tilde{g}$ with g on (M,g,S), determined by

$$\tilde{g}(x,y)=g(x,Sy)+g(Sx,y). \tag{2.5}$$

The fundamental tensor F of type $(0,3)$ and the 1-form θ are defined by

$$F(x,y,z)=(\nabla_x\tilde{g})(y,z),\quad \theta(x)=g^{ij}F(e_i,e_j,x), \tag{2.6}$$

and F has the property

$$F(x,z,y)=F(x,y,z). \tag{2.7}$$

The following necessary and sufficient conditions for S and also for $J=S^2$ to be parallel structures with respect to ∇ are established in [5].

Theorem 2.1. *The manifold (M,g,S) satisfies $\nabla S=0$ if and only if*

$$A_1=B_2-B_4,\quad A_2=B_1+B_3,\quad A_3=B_2+B_4,\quad A_4=B_3-B_1,$$

where $A_i=\dfrac{\partial A}{\partial x^i}$, $B_i=\dfrac{\partial B}{\partial x^i}$.

Theorem 2.2. *The structure S on (M,g,S) satisfies $\nabla S=0$ if and only if $\nabla J=0$, i.e. (M,g,J) is a Kähler manifold.*

3. The fundamental tensor *F* on *(M, g, S)*

In this section, we obtain a characteristic property of the tensor F on (M,g,S), which is an analogue of the property (2.4) of ∇J on (M,g,J). For this purpose we calculate the components of F.

Lemma 3.1. *The components* $F_{ijk} = F(e_i, e_j, e_k)$ *of the fundamental tensor* F *on the manifold* (M, g, S) *are given by*

$$
\begin{aligned}
F_{111} =& 2F_{124} = 2F_{313} = A_2 - A_4 - 2B_1,\\
F_{222} =& 2F_{424} = -2F_{213} = A_1 + A_3 - 2B_2,\\
F_{333} =& 2F_{113} = -2F_{324} = A_2 + A_4 - 2B_3,\\
F_{444} =& 2F_{224} = 2F_{413} = A_3 - A_1 - 2B_4,\\
F_{133} =& F_{311} = 0, \quad F_{244} = F_{422} = 0,\\
F_{233} =& F_{433} = -2F_{112} = -2F_{114} = -2F_{134} = 2F_{123} = B_2 + B_4 - A_3,\\
F_{122} =& F_{322} = -2F_{412} = -2F_{414} = -2F_{434} = 2F_{423} = B_1 + B_3 - A_2,\\
F_{211} =& -F_{411} = 2F_{312} = 2F_{314} = 2F_{334} = -2F_{323} = B_2 - B_4 - A_1,\\
F_{344} =& -F_{144} = 2F_{212} = 2F_{214} = 2F_{234} = -2F_{223} = B_3 - B_1 - A_4.
\end{aligned} \tag{3.1}
$$

Proof. The inverse matrix of (g_{ij}) has the form:

$$
(g^{ik}) = \frac{1}{D}\begin{pmatrix} A & -B & 0 & B\\ -B & A & -B & 0\\ 0 & -B & A & -B\\ B & 0 & -B & A \end{pmatrix}, \tag{3.2}
$$

where $D = A^2 - 2B^2$.

Using (2.1) and (2.3) we get that the matrix $\tilde{g}$, determined by (2.5), is of the type:

$$
(\tilde{g}_{ij}) = \begin{pmatrix} 2B & A & 0 & -A\\ A & 2B & A & 0\\ 0 & A & 2B & A\\ -A & 0 & A & 2B \end{pmatrix}. \tag{3.3}
$$

Due to (2.6) the components of F are $F_{ijk} = \nabla_i \tilde{g}_{jk}$. We apply to $\tilde{g}$ the following well-known formula for the covariant derivative of tensors:

$$
\nabla_i \tilde{g}_{jk} = \partial_i \tilde{g}_{jk} - \Gamma^a_{ij}\tilde{g}_{ak} - \Gamma^a_{ik}\tilde{g}_{aj}. \tag{3.4}
$$

Here Γ^s_{ij} are the Christoffel symbols of ∇. They are determined by

$$
2\Gamma^k_{ij} = g^{ak}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}). \tag{3.5}
$$

Then, with the help of (2.3), (2.6), (2.7), (3.2), (3.3) and (3.4) we calculate the components of F, given in (3.1). □

Immediately, we have the following

Corollary 3.2. *The components $\theta_k = g^{ij}F(e_i, e_j, e_k)$ of the 1-form θ on the manifold (M, g, S) are expressed by the equalities*

$$\begin{aligned}
\theta_1 &= \frac{2}{D}\big(A(A_2 - A_4 - 2B_1) - 2B(B_2 - B_4 - A_1)\big),\\
\theta_2 &= \frac{2}{D}\big(A(A_1 + A_3 - 2B_2) - 2B(B_1 + B_3 - A_2)\big),\\
\theta_3 &= \frac{2}{D}\big(A(A_2 + A_4 - 2B_3) + 2B(B_2 + B_4 - A_3)\big),\\
\theta_4 &= \frac{2}{D}\big(A(A_3 - A_1 - 2B_4) + 2B(B_3 - B_1 - A_4)\big).
\end{aligned} \tag{3.6}$$

Proof. The proof follows from (3.1) and (3.2) by direct computations. □

Corollary 3.3. *The components $\theta^*_k = g^{ij}F(e_i, Se_j, e_k)$ of the 1-form θ^* on the manifold (M, g, S) are expressed by the equalities*

$$\begin{aligned}
\theta^*_1 &= \frac{2}{D}\big(A(B_2 - B_4 - A_1) - B(A_2 - A_4 - 2B_1)\big),\\
\theta^*_2 &= \frac{2}{D}\big(A(B_1 + B_3 - A_2) - B(A_1 + A_3 - 2B_2)\big),\\
\theta^*_3 &= \frac{2}{D}\big(A(B_2 + B_4 - A_3) - B(A_2 + A_4 - 2B_3)\big),\\
\theta^*_4 &= \frac{2}{D}\big(A(B_3 - B_1 - A_4) - B(A_3 - A_1 - 2B_4)\big).
\end{aligned} \tag{3.7}$$

Proof. Using (2.1), (3.1) and (3.2), we find (3.7). □

Having in mind Lemma 3.1, Corollary 3.2 and Corollary 3.3 we get the next statements.

Theorem 3.4. *The fundamental tensor F on the manifold (M, g, S) satisfies the identity*

$$\begin{aligned}
F(x, y, z) =& \frac{1}{4}\{g(x, y)\theta(z) + g(x, z)\theta(y) + \big(g(Sx, y) + g(x, Sy)\big)\theta^*(z)\\
&+ \big(g(Sx, z) + g(x, Sz)\big)\theta^*(y)\}.
\end{aligned} \tag{3.8}$$

Proof. Using (2.1), (2.3), (3.1), (3.3), (3.6) and (3.7) we obtain

$$F_{kij} = \frac{1}{4}\big(g_{kj}\theta_i + g_{ki}\theta_j + \tilde{g}_{kj}\theta^*_i + \tilde{g}_{ki}\overset{*}{\theta}_j\big), \tag{3.9}$$

which is equivalent to (3.8). □

Remark 3.5. Comparing equalities in Theorem 2.1 and Lemma 3.1 we state that $F = 0$ if and only if the structure S is parallel with respect to ∇.

Theorem 3.6. *The fundamental tensor F on the manifold (M, g, S) has the property*

$$F(x, Jy, Jz) + F(y, Jz, Jx) + F(z, Jx, Jy) = 0, \tag{3.10}$$

where $J = S^2$.

Proof. Due to (2.1), (2.6), (2.7) and (3.1) we get that (3.10) holds true. □

Theorem 3.7. *Under the conformal transformation*

$$\overline{g}(x,y) = \alpha g(x,y), \tag{3.11}$$

where α *is a smooth positive function, the tensor* F *is transformed into the tensor*

$$\begin{aligned}\overline{F}(x,y,z) =& \frac{1}{4}\{\overline{g}(x,y)\overline{\theta}(z) + \overline{g}(x,z)\overline{\theta}(y) + \big(\overline{g}(Sx,y) + \overline{g}(x,Sy)\big)\overline{\theta}^*(z) \\ &+ \big(\overline{g}(Sx,z) + \overline{g}(x,Sz)\big)\overline{\theta}^*(y)\}\end{aligned} \tag{3.12}$$

with $\overline{\theta} = \theta + \dfrac{2}{\alpha}\mathrm{d}\alpha \circ (S - S^3)$ *and* $\overline{\theta}^* = \theta^* - \dfrac{2}{\alpha}\mathrm{d}\alpha$.

Proof. The inverse matrix of $(\tilde{g}_{ij})$ has the form

$$(\tilde{g}^{ik}) = \frac{1}{2D}\begin{pmatrix} -2B & A & 0 & -A \\ A & -2B & A & 0 \\ 0 & A & -2B & A \\ -A & 0 & A & -2B \end{pmatrix}. \tag{3.13}$$

Bearing in mind (2.3), (3.2), (3.3) and (3.13), we get

$$\tilde{g}_{ij}g^{is} = \Phi_j^s\ , \quad g_{ij}\tilde{g}^{is} = \frac{1}{2}\Phi_j^s, \tag{3.14}$$

where

$$(\Phi_j^s) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}. \tag{3.15}$$

Because of (2.1) and (3.15) we have $\Phi = S - S^3$.

Now, from (3.6), (3.7) and (3.15), we find

$$\theta_i^* = -\frac{1}{2}\Phi_i^s\theta_s. \tag{3.16}$$

According to the transformation (3.11), the components of the tensor $\overline{F}$ are $\overline{F}_{ijk} = \overline{\nabla}_i\overline{\tilde{g}}_{jk}$, where $\overline{\tilde{g}} = \alpha\tilde{g}$ and $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$. Therefore, it follows

$$\overline{\nabla}\ \overline{\tilde{g}} = \alpha\overline{\nabla}\tilde{g} + \tilde{g}\overline{\nabla}\alpha. \tag{3.17}$$

From the Christoffel formulas (3.5) and

$$2\overline{\Gamma}_{ij}^k = \overline{g}^{ks}(\partial_i\overline{g}_{sj} + \partial_j\overline{g}_{si} - \partial_s\overline{g}_{ij}),$$

and due to (3.11) we get

$$\overline{\Gamma}_{ij}^{k}=\Gamma_{ij}^{k}+\frac{1}{2\alpha}(\delta_j^k\alpha_i+\delta_i^k\alpha_j-g_{ij}g^{ks}\alpha_s),\quad \alpha_s=\frac{\partial\alpha}{\partial x^s}.$$

Then, applying (3.4) to $\overline{\nabla}\tilde{g}$ and using (3.14), we obtain

$$\begin{aligned}\overline{\nabla}_k\tilde{g}_{ji}=\nabla_k\tilde{g}_{ji}&-\frac{1}{2\alpha}(\tilde{g}_{ji}\alpha_k+\tilde{g}_{ik}\alpha_j-g_{kj}\Phi_i^s\alpha_s)\\&-\frac{1}{2\alpha}(\tilde{g}_{kj}\alpha_i+\tilde{g}_{ij}\alpha_k-g_{ik}\Phi_j^s\alpha_s).\end{aligned}\tag{3.18}$$

Substituting (3.18) into (3.17), and taking into account (3.9), (3.14), (3.15) and (3.16), we get

$$\begin{aligned}\overline{\nabla}_k\bar{\tilde{g}}_{ji}=&\frac{1}{4}\{\alpha g_{kj}\big(\theta_i+\frac{2\alpha_s}{\alpha}\Phi_i^s\big)+\alpha g_{ki}\big(\theta_j+\frac{2\alpha_s}{\alpha}\Phi_j^s\big)\\&+\alpha\tilde{g}_{kj}\big(\theta_i^*-\frac{2\alpha_i}{\alpha}\big)+\alpha\tilde{g}_{ki}\big(\theta_j^*-\frac{2\alpha_j}{\alpha}\big)\},\end{aligned}$$

which implies

$$\begin{aligned}\overline{F}_{kji}=&\frac{1}{4}\big(\overline{g}_{kj}\overline{\theta}_i+\overline{g}_{ki}\overline{\theta}_j+\bar{\tilde{g}}_{kj}\overline{\theta}_i^*+\bar{\tilde{g}}_{ki}\overline{\theta}_j^*\big),\\&\overline{\theta}_i=\theta_i+\frac{2}{\alpha}\Phi_i^s\alpha_s,\quad \overline{\theta}_i^*=-\frac{1}{2}\Phi_i^s\overline{\theta}_s.\end{aligned}$$

Thus, for $(M,\overline{g},S)$, the identity (3.12) is valid . □

Remark 3.8. According to Theorem 3.7, we can say that (M,g,S) and $(M,\overline{g},S)$ belong to classes of the same type, defined by the equality (3.8) for the corresponding metric.

Immediately, from (2.5), (2.6), (3.12) and (3.14) it follows

Corollary 3.9. *If $F=0$ holds, then it is valid*

$$\begin{aligned}\overline{F}(x,y,z)=&\frac{1}{2}\{g(x,y)\alpha(\Phi z)+g(x,z)\alpha(\Phi y)-\tilde{g}(x,y)\alpha(z)\\&-\tilde{g}(x,z)\alpha(y)\}.\end{aligned}\tag{3.19}$$

Next, we obtain

Corollary 3.10. *If $F=0$ holds, then $\overline{F}$ vanishes if and only if α is a constant.*

Proof. The local form of (3.19) is

$$\overline{F}_{kij}=\frac{1}{2}\big(g_{kj}\Phi_i^s\alpha_s+g_{ki}\Phi_j^s\alpha_s-\tilde{g}_{kj}\alpha_i-\tilde{g}_{ki}\alpha_j\big).\tag{3.20}$$

Let the tensor $\overline{F}$ vanish. Hence equality (3.20) yields

$$g_{kj}\Phi_i^s\alpha_s+g_{ki}\Phi_j^s\alpha_s-\tilde{g}_{kj}\alpha_i-\tilde{g}_{ki}\alpha_j=0.$$

Contracting by g^{kj} in the latter equality, and using (3.14) and (3.15), we find $\Phi_i^s\alpha_s=0$, which implies $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$, i.e. α is a constant.

Vice versa. If α is a constant, then (3.20) implies $\overline{F}=0$. □

4. Some curvature properties of *(M, g, S)*

It is well-known, that the curvature tensor R of ∇ is defined by

$$R(x,y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

The tensor of type $(0,4)$ associated with R is defined as follows:

$$R(x,y,z,u) = g(R(x,y)z,u).$$

The Ricci tensor ρ and the scalar curvature τ with respect to g are as usually:

$$\rho(y,z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j). \tag{4.1}$$

In this section we investigate some curvature properties of (M, g, S), corresponding to the metric g and to the associated metric $\tilde{g}$.

Let $\tilde{\Gamma}$ be the Christoffel symbols of $\tilde{g}$ and $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{g}$. Let $\tilde{R}$ be the curvature tensor of $\tilde{\nabla}$. The Ricci tensor $\tilde{\rho}$ and the scalar curvature $\tilde{\tau}$ with respect to $\tilde{g}$ are given by

$$\tilde{\rho}(y,z) = \tilde{g}^{ij} \tilde{R}(e_i, y, z, e_j), \ \tilde{\tau} = \tilde{g}^{ij} \tilde{\rho}(e_i, e_j). \tag{4.2}$$

Let us denote

$$\tau^* = \tilde{g}^{ij} \rho(e_i, e_j), \quad \tilde{\tau}^* = g^{ij} \tilde{\rho}(e_i, e_j). \tag{4.3}$$

Therefore we establish the following

Theorem 4.1. *Let $\tilde{g}$ be the associated metric with g on (M, g, S). For the Ricci tensors ρ and $\tilde{\rho}$ and for the scalar quantities τ, τ^*, $\tilde{\tau}$ and $\tilde{\tau}^*$ the following relation is valid:*

$$\tilde{\rho}(x,y) = \rho(x,y) + \frac{1}{4}(\tilde{\tau}^* - \tau) g(x,y) + \frac{1}{4}(\tilde{\tau} - \tau^*) \tilde{g}(x,y). \tag{4.4}$$

Proof. From (3.4), applying the Christoffel formulas (3.5) to Γ and also to $\tilde{\Gamma}$, we obtain

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{1}{2} \tilde{g}^{ks} (\nabla_i \tilde{g}_{js} + \nabla_j \tilde{g}_{is} - \nabla_s \tilde{g}_{ij}).$$

Substituting (3.9) into the above equality, we get

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{1}{4} \tilde{g}^{ks} (g_{ij} \theta_s + \tilde{g}_{ij} \theta^*_s). \tag{4.5}$$

Using (3.14), (3.15) and (3.16) we find

$$\tilde{g}^{sk} \theta_s = -\theta^{*k}, \quad \tilde{g}^{sk} \theta^*_s = -\frac{1}{2} \theta^k, \quad \tilde{g}_{sk} \theta^s = -2\theta^*_k, \quad \tilde{g}_{sk} \theta^{*s} = -\theta_k. \tag{4.6}$$

Bearing in mind (4.5), the first and the second equality of (4.6), we calculate the components of the tensor $\mathrm{T} = \tilde{\Gamma} - \Gamma$ of the affine deformation. They are as follows:

$$T_{ij}^{k} = -\frac{1}{4}\big(g_{ij}\theta^{*k} + \frac{1}{2}\tilde{g}_{ij}\theta^{k}\big). \tag{4.7}$$

For the components of the curvature tensors $\tilde{R}$ and R, it is well-known the relation

$$\tilde{R}_{ijs}^{k} = R_{ijs}^{k} + \nabla_j T_{is}^{k} - \nabla_s T_{ij}^{k} + T_{is}^{a}T_{aj}^{k} - T_{ij}^{a}T_{as}^{k}.$$

Then, taking into account (3.9), (3.16), (4.6) and (4.7), we calculate

$$\begin{aligned}\tilde{R}_{ijs}^{k} =& R_{ijs}^{k} - \frac{1}{4}g_{is}(\nabla_j\theta^{*k} - \frac{1}{4}\theta_j^{*}\theta^{*k}) + \frac{1}{4}g_{ij}(\nabla_s\theta^{*k} - \frac{1}{4}\theta_s^{*}\theta^{*k})\\ &- \frac{1}{8}\tilde{g}_{is}(\nabla_j\theta^{k} - \frac{1}{4}\theta_j\theta^{*k}) + \frac{1}{8}\tilde{g}_{ij}(\nabla_s\theta^{k} - \frac{1}{4}\theta_s\theta^{*k}).\end{aligned}$$

By contracting $k = s$ in the latter equality, and having in mind (3.9), (3.14), (3.16), (4.1), (4.2) and (4.6), we get

$$\tilde{\rho}_{ij} = \rho_{ij} + \frac{1}{4}g_{ij}\nabla_s\theta^{*s} + \frac{1}{4}\tilde{g}_{ij}\nabla_s\theta^{s}. \tag{4.8}$$

Due to (3.14), (4.1), (4.2), (4.3) and (4.8) we obtain

$$\tilde{\rho}_{ij} = \rho_{ij} + \frac{1}{4}(\tilde{\tau}^{*} - \tau)g_{ij} + \frac{1}{4}(\tilde{\tau} - \tau^{*})\tilde{g}_{ij},$$

which is a local form of (4.4). $\square$

Further, we use the following statements for a special basis of T_pM on (M, g, S) that are established in [5].

(i) A basis of type $\{S^3x, S^2x, Sx, x\}$ of T_pM exists and it is called an *S-basis*. In this case we say that *the vector x induces an S-basis of T_pM*.

(ii) If a vector x induces an S-basis and φ is the angle between x and Sx, then

$$g(x, Sx) = g(x, x)\cos\varphi, \qquad \tilde{g}(x, x) = 2g(x, x)\cos\varphi, \tag{4.9}$$

and $\frac{\pi}{4} < \varphi < \frac{3\pi}{4}$.

(iii) An orthogonal *S-basis* of T_pM exists.

Now, we recall that the Ricci curvature, with respect to g, in the direction of a nonzero vector x is the value

$$r(x) = \frac{\rho(x, x)}{g(x, x)}. \tag{4.10}$$

Due to Theorem 4.1 we establish the following

Corollary 4.2. *Let a vector x induce an S-basis of T_pM and let φ be the angle between x and Sx. If r and $\tilde{r}$ are the Ricci curvatures in the direction of x with respect to the metrics g and $\tilde{g}$, then*

$$\tilde{r}(x) = \frac{1}{2\cos\varphi} r(x) + \frac{1}{8\cos\varphi}(\tilde{\tau}^* - \tau) + \frac{1}{4}(\tilde{\tau} - \tau^*), \quad \varphi \neq \frac{\pi}{2}. \tag{4.11}$$

Proof. The proof follows directly from (4.4), (4.9) and (4.10). □

In [5], a Riemannian manifold (M, g, S) is called almost Einstein if the metrics g and $\tilde{g}$ satisfy

$$\rho(x, y) = \beta g(x, y) + \gamma \tilde{g}(x, y), \tag{4.12}$$

where β and γ are smooth functions on M.

It is known that a Riemannian manifold (M, g) is called Einstein if the metric g satisfies

$$\rho(x, y) = \beta g(x, y). \tag{4.13}$$

Proposition 4.3. *Let the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ be a locally flat connection on the manifold (M, g, S). Then the following statements are valid.*

(i) (M, g, S) is an almost Einstein manifold, and the Ricci tensor ρ has the form

$$\rho(x, y) = \frac{\tau}{4} g(x, y) + \frac{\tau^*}{4} \tilde{g}(x, y). \tag{4.14}$$

(ii) If a vector x induces an S-basis, then the Ricci curvatures in the direction of the basis vectors are

$$r(x) = r(Sx) = r(S^2x) = r(S^3x) = \frac{\tau}{4} + \frac{\tau^*}{2}\cos\varphi, \tag{4.15}$$

where $\varphi = \angle(x, Sx)$.

Proof. If $\tilde{\nabla}$ is a locally flat connection, then $\tilde{R} = 0$. From (4.2) and (4.3) it follows $\tilde{\rho} = 0$ and $\tilde{\tau} = \tilde{\tau}^* = 0$. Hence (4.4) implies (4.14). Therefore, according to (4.12), we have that (M, g, S) is an almost Einstein manifold.

Since ρ is given by (4.14), using (2.2) and (2.5), we obtain

$$\begin{aligned} \rho(x, x) = & \rho(Sx, Sx) = \rho(S^2x, S^2x) = \rho(S^3x, S^3x) \\ & = \frac{\tau}{4} g(x, x) + \frac{\tau^*}{4}\tilde{g}(x, x). \end{aligned} \tag{4.16}$$

Let a vector x induce an S-basis. Hence equalities (4.9), (4.10) and (4.16) imply (4.15). □

Corollary 4.4. *Let (M, g, S) be an Einstein manifold. If a vector x induces an S-basis, then the Ricci curvatures in the direction of the basis vectors are*

$$r(x) = r(Sx) = r(S^2x) = r(S^3x) = \frac{\tau}{4}.$$

Proof. By comparing (4.14) and (4.13) we have that τ^* vanishes. Thus the above equalities follow directly by substituting $\tau^* = 0$ into (4.15). $\square$

In a similar way to Proposition 4.3 we prove the next

Proposition 4.5. *Let the Levi-Civita connection ∇ of g be a locally flat connection on the manifold $(M, \tilde{g}, S)$. Then the following statements are valid.*

(i) $(M, \tilde{g}, S)$ is an almost Einstein manifold and the Ricci tensor $\tilde{\rho}$ has the form

$$\tilde{\rho}(x, y) = \frac{\tilde{\tau}}{4}\tilde{g}(x, y) + \frac{\tilde{\tau}^*}{4}g(x, y).$$

(ii) If a vector x induces an S-basis, then the Ricci curvatures in the direction of the basis vectors are

$$\tilde{r}(x) = \tilde{r}(Sx) = \tilde{r}(S^2x) = \tilde{r}(S^3x) = \frac{\tilde{\tau}}{4} + \frac{\tilde{\tau}^*}{8\cos\varphi}, \quad \varphi \neq \frac{\pi}{2}.$$

5. A locally conformal Kähler manifold *(M, g, J)*

The fundamental Kähler form of the structure (g, J) on an almost complex manifold (M, g, J) is determined by

$$J(x, y) = g(x, Jy) \tag{5.1}$$

and it is skew-symmetric, i.e. $J(x, y) = -J(y, x)$ ([6]).

In this section we consider a Hermitian manifold (M, g, J) with a complex structure $J = S^2$.

Lemma 5.1. *The nonzero components $\nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k)$ of the fundamental tensor ∇J on the manifold (M, g, J) are given by*

$$\begin{array}{l}
\nabla_3 J_{12} = -\nabla_3 J_{34} = \nabla_1 J_{23} = -\nabla_1 J_{14} = \frac{1}{2}(B_1 + B_3 - A_2),\\
\nabla_1 J_{34} = -\nabla_1 J_{12} = \nabla_3 J_{23} = -\nabla_3 J_{14} = \frac{1}{2}(A_4 + B_1 - B_3),\\
\nabla_2 J_{34} = -\nabla_2 J_{12} = \nabla_4 J_{23} = -\nabla_4 J_{14} = \frac{1}{2}(B_2 + B_4 - A_3),\\
\nabla_4 J_{12} = -\nabla_4 J_{34} = \nabla_2 J_{23} = -\nabla_2 J_{14} = \frac{1}{2}(A_1 + B_4 - B_2).
\end{array} \tag{5.2}$$

Proof. Because of (2.1), (2.3) and (5.1), we get that the matrix of the fundamental Kähler form is of the type:

$$(J_{ik}) = \begin{pmatrix} 0 & B & A & B \\ -B & 0 & B & A \\ -A & -B & 0 & B \\ -B & -A & -B & 0 \end{pmatrix}. \tag{5.3}$$

Applying the Christoffel symbols Γ, obtained by (2.3), (3.2) and (3.5), and the components of the matrix (5.3) to equality

$$\nabla_i J_{jk} = \partial_i J_{jk} - \Gamma^a_{ij} J_{ak} - \Gamma^a_{ik} J_{aj},$$

we calculate the nonzero components of ∇J, given in (5.2). $\square$

Immediately, we have the following

Corollary 5.2. *The components $\omega_k = g^{ij}g((\nabla_{e_i}J)e_j, e_k)$ of the 1-form ω on the manifold (M, g, J) are expressed by the equalities*

$$\begin{aligned}
\omega_1 &= \frac{1}{D}\big(A(B_4 + B_2 - A_3) + B(2B_3 - A_2 - A_4)\big),\\
\omega_2 &= \frac{1}{D}\big(A(B_3 - B_1 - A_4) + B(2B_4 + A_1 - A_3)\big),\\
\omega_3 &= \frac{1}{D}\big(A(A_1 + B_4 - B_2) + B(-2B_1 + A_2 - A_4)\big),\\
\omega_4 &= \frac{1}{D}\big(A(A_2 - B_1 - B_3) + B(-2B_2 + A_1 + A_3)\big),
\end{aligned} \tag{5.4}$$

Proof. The equalities (5.4) follow from (3.2) and (5.2) by direct computations. □

Due to Lemma 5.1 and Corollary 5.2 we establish the following

Theorem 5.3. *The manifold (M, g, J) is a locally conformal Kähler manifold.*

Proof. Using (2.1), (2.3), (5.3), (5.4) and Lemma 5.1 we obtain

$$\nabla_k J_{ij} = \frac{1}{2}\big(g_{ki}\omega_j - g_{kj}\omega_i + J_{ki}\tilde{\omega}_j - J_{kj}\tilde{\omega}_i\big), \quad \tilde{\omega}_i = J_i^a\omega_a. \tag{5.5}$$

The latter identity is the local form of (2.4), which is a defining condition of a locally conformal Kähler manifold. □

6. A Lie group with a structure *(g, S)*

Let G be a 4-dimensional real connected Lie group. Let $\mathfrak{g}$ be the corresponding Lie algebra with a basis $\{e_1, e_2, e_3, e_4\}$ of left invariant vector fields. We introduce a skew-circulant structure S and a metric g as follows:

$$Se_1 = -e_4, \quad Se_2 = e_1, \quad Se_3 = e_2, \quad Se_4 = e_3; \tag{6.1}$$

$$g(e_i, e_j) = \delta_{ij}, \tag{6.2}$$

where δ_{ij} is the Kronecker delta.

Consequently the used basis $\{e_i\}$ is an orthonormal S-basis. Obviously, (2.1) and (2.2) are valid and (g, S) is a structure of the considered type. We denote the corresponding manifold by (G, g, S). Then the associated manifold is (G, g, J), where J satisfies

$$Je_1 = -e_3, \quad Je_2 = -e_4, \quad Je_3 = e_1, \quad Je_4 = e_2. \tag{6.3}$$

The real four-dimensional indecomposable Lie algebras are classified by Mubarakzyanov ([10]). This scheme seems to be the most popular (see [2] and the references therein). We pay attention to the class $\{\mathfrak{g}_{4,5}\}$, which represents an indecomposable Lie algebra, depending on two real parameters a and

b. Actually, it induces a family of manifolds whose properties are functions of a and b.

According to the definition of the class $\{\mathfrak{g}_{4,5}\}$, we have that the nonzero brackets are as follows ([2]):

$$[e_1, e_4] = e_1,\ [e_2, e_4] = ae_2,\ [e_3, e_4] = be_3, \quad -1 \leq b \leq a \leq 1,\ ab \neq 0. \tag{6.4}$$

The well-known Koszul formula implies

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g([e_k, e_j], e_i)$$

and, using (6.2) and (6.4), we obtain

$$\begin{array}{lll} \nabla_{e_1} e_1 = -e_4, & \nabla_{e_1} e_4 = e_1, & \nabla_{e_2} e_2 = -ae_4, \\ \nabla_{e_2} e_4 = ae_2, & \nabla_{e_3} e_4 = be_3, & \nabla_{e_3} e_3 = -be_4. \end{array} \tag{6.5}$$

Furthermore, with the help of the above equalities we compute the components of the tensor F on (G, g, S) and the components of the tensor ∇J on (G, g, J). We find conditions under which F satisfies (3.8) and ∇J satisfies (2.4).

Proposition 6.1. *If $\mathfrak{g}$ belongs to $\{\mathfrak{g}_{4,5}\}$, then the fundamental tensor F on (G, g, S) satisfies the property (3.8) if and only if the condition $a = b = 1$ holds.*

Proof. Bearing in mind (2.5), (2.6), (6.1), (6.2) and (6.5) we get the components F_{ijk} of F, θ_i of θ and θ^*_i of θ^* with respect to the basis $\{e_i\}$. The nonzero of them are the following:

$$\begin{array}{ll} F_{124} = -F_{113} = -F_{134} = -1, & F_{111} = -F_{144} = -2, \\ F_{313} = F_{332} = F_{324} = -b, & F_{333} = -F_{344} = 2b, \\ F_{212} = F_{214} = -F_{223} = F_{234} = -a. & \end{array} \tag{6.6}$$

$$\begin{array}{ll} \theta_1 = -2 - a - b, & \theta_3 = 2a + b + 1, \\ \theta^*_2 = \frac{1}{2}(1 - b), & \theta^*_4 = -\frac{1}{2}(2a + 3b + 3). \end{array} \tag{6.7}$$

By equalities (2.5), (6.1) and (6.2), we find

$$\begin{array}{l} \tilde{g}(e_1, e_1) = \tilde{g}(e_2, e_2) = \tilde{g}(e_3, e_3) = \tilde{g}(e_4, e_4) = 0, \\ \tilde{g}(e_1, e_3) = \tilde{g}(e_2, e_4) = 0, \\ \tilde{g}(e_1, e_2) = \tilde{g}(e_2, e_3) = \tilde{g}(e_3, e_4) = -\tilde{g}(e_1, e_4) = 1. \end{array} \tag{6.8}$$

Hence (6.1), (6.2), (6.6), (6.7) and (6.8) imply that the condition (3.9) holds if and only if $a = b = 1$. $\square$

Proposition 6.2. *If $\mathfrak{g}$ belongs to $\{\mathfrak{g}_{4,5}\}$, then (G, g, J) belongs to the class of locally conformal Kähler manifolds if and only if the condition $a = b = 1$ holds.*

Proof. Bearing in mind (5.1), (6.2), (6.3) and (6.5) we obtain the components $\nabla_i J_{jk}$ of ∇J and ω_i of ω with respect to the basis $\{e_i\}$. The nonzero of them are the following:

$$\nabla_1 J_{12} = -\nabla_1 J_{34} = 1, \quad \nabla_3 J_{32} = \nabla_3 J_{14} = b, \quad \omega_2 = b + 1. \tag{6.9}$$

By equalities (5.1), (6.1), (6.2) and (6.9), we get that the condition (5.5) holds if and only if $a = b = 1$. □

Remark 6.3. Obviously, if $\mathfrak{g}$ is in $\{\mathfrak{g}_{4,5}\}$, then (G, g, J) is not a Kähler manifold.

By virtue of Proposition 6.1 and Proposition 6.2, we immediately have the following

Corollary 6.4. *If $\mathfrak{g}$ belongs to $\{\mathfrak{g}_{4,5}\}$, then the fundamental tensor F on (G, g, S) satisfies (3.8) if and only if (G, g, J) belongs to the class of locally conformal Kähler manifolds.*

Next we get

Proposition 6.5. *Let (G, g, S) be a manifold with a Lie algebra $\mathfrak{g}$ from the class $\{\mathfrak{g}_{4,5}\}$. If $a = b = 1$ are valid, then (G, g, S) is a non-flat Einstein manifold with a negative scalar curvature $\tau = -12$.*

Proof. We calculate the components R_{ijks} of the curvature tensor R with respect to $\{e_i\}$, having in mind the symmetries of R and the condition $a = b = 1$. The nonzero of them are

$$R_{1212} = R_{1414} = R_{2323} = R_{3434} = R_{1313} = R_{2424} = 1. \tag{6.10}$$

Using (4.1) and (6.10), we compute the components of ρ and the value of τ. The nonzero of them are as follows:

$$\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} = -3, \qquad \tau = -12.$$

Then, from (6.2), we get $\rho = \frac{\tau}{4} g$. Consequently, due to (4.13), the manifold (G, g, S) is Einstein.

□

Remark 6.6. Lie groups with a Lie algebra in $\{\mathfrak{g}_{4,5}\}$ are studied in [4] as an example of 4-dimensional Riemannian manifolds with circulant structures.

An example of a locally conformal Kähler manifold constructed on a Lie group with a Lie algebra in $\{\mathfrak{g}_{4,5}\}$ is considered in [8]. The metric of the manifold is indefinite but the condition $a = b = 1$ also exists.

The example of a 4-dimensional Riemannian manifold with skew-circulant structures, constructed in this section, has similar properties to the above examples.

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