ПЛОВДИВСКИ УНИВЕРСИТЕТ “ПАИСИЙ ХИЛЕНДАРСКИ”, БЪЛГАРИЯ
НАУЧНИ ТРУДОВЕ, ТОМ 38, КН. 3, 2011 – МАТЕМАТИКА
PLOVDIV UNIVERSITY “PAISSII HILENDARSKI”, BULGARIA
SCIENTIFIC WORKS, VOL. 38, BOOK 3, 2011 – MATHEMATICS

ON A THREE-DIMENSIONAL RIEMANNIAN MANIFOLD WITH AN ADDITIONAL STRUCTURE

Georgi Dzhelepov, Iva Dokuzova, Dimitar Razpopov

Abstract. We consider a 3-dimensional Riemannian manifold M with a metric tensor g, and affinors q and S. We note that the local coordinates of these three tensors are circulant matrices. We have that the third degree of q is the identity and q is compatible with g. We discuss the sectional curvatures in case when q is parallel with respect to the connection of g.

Key words: Riemannian metric, affinor structure, sectional curvatures
Mathematics Subject Classification 2000: 3C05, 53B20

1. Introduction

Many papers in the differential geometry have been dedicated on the problems in the differential manifolds admitting an additional affinor structure f. In the most of them f satisfies some identities of the second degree $f^2 = id$, or $f^2 = -id$. We note two papers [7], [8] where f satisfies the equation of the third degree $f^3 + f = 0$.

Let a differential manifold admit an affine connection ∇ and an affinor structure f. If ∇f satisfies some equation there follows an useful curvature identity. Such identities and assertions were obtained in the almost Hermitian geometry in [2]. Analogous results have been discussed for the almost complex manifolds with Norden metric in [1], [3] and [4], and for the almost contact manifolds with B-metric in [5] and [6].

In the present paper we are interested in a three-dimensional Riemannian manifold M with an affinor structure q. The structure satisfies the identity

$q^3 = id$, $q \neq \pm id$ and q is compatible with the Riemannian metric of M. Moreover, we suppose the local coordinates of these structures are circulant. We search conditions the structure q to be parallel with respect to the Riemannian connection ∇ of g (i.e. $\nabla q = 0$). We get some curvature identities in this case.

2. Preliminaries

It is known from the linear algebra, that the set of circulant matrices of type $(n \times n)$ is a commutative group. In the present paper we use four circulant matrices of type (3×3) for geometrical considerations, as follows:

$$(g_{ij}) = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, \quad A > B > 0, \tag{1}$$

where $A = A(X^1, X^2, X^3)$, $B = B(X^1, X^2, X^3)$; and $X^1, X^2, X^3 \in R$.

$$(g^{ij}) = \frac{1}{D}\begin{pmatrix} A+B & -B & -B \\ -B & A+B & -B \\ -B & -B & A+B \end{pmatrix}, \quad D = (A-B)(A+2B), \tag{2}$$

$$\left(q_i^{\cdot j}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3}$$

$$\left(S_i^{\cdot j}\right) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \tag{4}$$

We choose the form in (3) of the matrix q because of the next assertion:

Lemma 1. *Let (m_{ij}), $i, j = 1, 2, 3$ be a circulant non-degenerate matrix and its third degree is the unit matrix.*

Then (m_{ij}) has one of the following forms:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5}$$

Proof. If $\left(m_{ij}\right)$ has the form

$$\left(m_{ij}\right)=\begin{pmatrix} a & b & c\\ c & a & b\\ b & c & a\end{pmatrix},$$

then from the condition $\left(m_{ij}\right)^3=E$ (E is the unit matrix) we get the system

$$\begin{aligned}&a^3+b^3+c^3+6\,a\,b\,c=1\\ &a^2b+ac^2+b^2c=0\\ &ab^2+ca^2+c^2b=0.\end{aligned}$$

The all solutions of this system are (5).

3. A Parallel Structure

Let M be a 3-dimensional Riemannian manifold and $\{e_1,\,e_2,\,e_3\}$ be a basis of the tangent space T_pM at a point $p\big(X^1,\,X^2,\,X^3\big)\in M$. Let g be a metric tensor and q be an affinor, which local coordinates are given in (1) and (3), respectively. Let A and B from (1) be smooth functions of a point p in some coordinate neighborhood $F\subset R^3$. We will use the notation $\Phi_i=\dfrac{\partial\Phi}{\partial X^i}$ for every smooth function Φ, defined in F. We verify that the following identities are true

$$q^3=E;\quad g(qx,qy)=g(x,y),\quad x,\ y\in\chi M, \tag{6}$$

as well as

$$g_{is}\,g^{js}=\delta_i^j. \tag{7}$$

Let ∇ be the Riemannian connection of g and Γ_{ij}^s be the Christoffel symbols of ∇. It is well known the next formula

$$2\Gamma_{ij}^s=g^{as}\left(\partial_ig_{aj}+\partial_jg_{ai}-\partial_ag_{ij}\right). \tag{8}$$

Using (1), (2), (7), (8), after long computations we get the next equalities:

$$
\begin{aligned}
\Gamma^i_{ii} &= \frac{1}{2D}\left((A+B)A_i - B(4B_i - A_j - A_k)\right), \\
\Gamma^k_{ii} &= \frac{1}{2D}\left((A+B)(2B_i - A_k) - B(2B_i - A_j + A_i)\right), \\
\Gamma^i_{ij} &= \frac{1}{2D}\left((A+B)A_j - B(-B_k + B_i + B_j + A_i)\right), \\
\Gamma^k_{ij} &= \frac{1}{2D}\left((A+B)(-B_k + B_i + B_j) - B(A_i + A_j)\right),
\end{aligned}
\tag{9}
$$

where $i \neq j \neq k$ and $i = 1, 2, 3$, $j = 1, 2, 3$, $k = 1, 2, 3$.

Theorem 1. *Let M be the Riemannian manifold, supplied with a metric tensor g, and affinors q and S, defined by (1), (3) and (4), respectively. The structure q is parallel with respect to the Riemannian connection ∇ of g, if and only if*

$$grad\, A = grad\, B.S. \tag{10}$$

Proof.
a) Let q be a parallel structure with respect to ∇, i.e.

$$\nabla q = 0. \tag{11}$$

In terms of the local coordinates, the last equation implies

$$\nabla_i q^s_j = \partial_i q^s_j + \Gamma^s_{ia} q^a_j - \Gamma^a_{ij} q^s_a = 0,$$

which, by virtue of (3), is equivalent to

$$\Gamma^s_{ia} q^{.a}_{j} = \Gamma^a_{ij} q^{.s}_{a}. \tag{12}$$

Using (3), (9) and (12), we get 18 equations which all imply (10).

b) Vice versa, let (10) be valid. Then from (9) we get

$$
\begin{aligned}
\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^3_{22} = \Gamma^1_{23} = \Gamma^2_{33} &= \frac{1}{2D}(AA_1 + B(-3B_1 + B_2 + B_3)), \\
\Gamma^3_{11} = \Gamma^1_{12} = \Gamma^2_{13} = \Gamma^2_{22} = \Gamma^3_{23} = \Gamma^1_{33} &= \frac{1}{2D}(AA_2 + B(B_1 - 3B_2 + B_3)), \\
\Gamma^2_{11} = \Gamma^3_{12} = \Gamma^1_{13} = \Gamma^1_{22} = \Gamma^2_{23} = \Gamma^3_{33} &= \frac{1}{2D}(AA_3 + B(B_1 + B_2 - 3B_3)).
\end{aligned}
$$

Now, we can verify that (12) is valid. That means $\nabla_i q_j^s = 0$, i.e. $\nabla q = 0$. □

Remark. *In fact (10) is a system of three partial differential equations for the functions A and B. Let $p(X^1, X^2, X^3)$ be a point in M. We assume $B = B(p)$ as a known function and then we can say that (10) has a solution. Particularly, we give a simple (but non-trivial example) for both functions, satisfying (10), as follows $A = (X^1)^2 + (X^2)^2 + (X^3)^2$; $B = X^1X^2 + X^1X^3 + X^2X^3$, where $A > B > 0$.*

4. Sectional Curvatures

Let M be the Riemannian manifold with a metric tensor g and a structure q, defined by (1) and (3), respectively. Let R be the curvature tensor field of ∇, i.e $R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. We consider the associated tensor field R of type $(0,4)$, defined by the condition

$$R(x,y,z,u) = g\big(R(x,y)z,u\big), \qquad x,\ y,\ z,\ u \in \chi M.$$

Theorem 2. *If M is the Riemannian manifold with a metric tensor g and a parallel structure q, defined by (1) and (3), respectively, then the curvature tensor R of g satisfies the identity:*

$$R(x,y,q^2z,u) = R(x,y,z,qu), \qquad x,y,z,u \in \chi M. \tag{13}$$

Proof. In terms of the local coordinates (11) implies

$$R^l_{sji}\, q_k^{.s} = R^s_{kji}\, q_s^{.l}. \tag{14}$$

Using (3), we verify $q^{i}_{.j} = q_a^{.i}\, q_j^{.a}$ and then from (1), (2) and (14) we obtain (13).

□

Let p be a point in M and x, y be two linearly independent vectors on T_pM. It is known that the quantity

$$\mu(L;p) = \frac{R(x,y,x,y)}{g(x,x)g(y,y) - g^2(x,y)} \tag{15}$$

is the sectional curvature of 2-plane $L = \{x,y\}$.

Let p be a point in M and $x = (x^1,\, x^2,\, x^3)$ be a vector in T_pM. The vectors $x,\ qx,\ q^2x$ are linearly independent, when

$$3x^1x^2x^3 \neq (x^1)^3 + (x^2)^3 + (x^3)^3. \tag{16}$$

Then we define 2-planes $L_1 = \{x, qx\}$, $L_2 = \{qx, q^2x\}$ and $L_3 = \{q^2x, x\}$ and we prove the following

Theorem 3. *Let M be the Riemannian manifold with a metric tensor g and a parallel structure q, defined by (1) and (3), respectively. Let p be a point in M and x be an arbitrary vector in T_pM satisfying (16). Then the sectional curvatures of 2-planes $L_1 = \{x, qx\}$, $L_2 = \{qx, q^2x\}$, $L_3 = \{q^2x, x\}$ are equal.*

Proof. From (13) we obtain

$$R(x,\, y,\, z,\, u) = R(x,\, y,\, qz,\, qu) = R(x,\, y,\, q^2z,\, q^2u). \tag{17}$$

In (17) we set the following substitutions: a) $z = x$, $y = u = qx$; b) $x \sim qx$, $z = qx$, $y = u = q^2x$; c) $x \sim q^2x$, $z = q^2x$, $y = u = x$. Comparing the obtained results, we get

$$\begin{aligned} R(x,\, qx,\, q^2x,\, x) &= R(x,\, qx,\, qx,\, q^2x) \\ &= R(q^2x,\, x,\, qx,\, q^2x) \\ &= R(x,\, qx,\, x,\, qx) \end{aligned} \tag{18}$$

and

$$R(x,\, qx,\, x,\, qx) = R(qx,\, q^2x,\, qx,\, q^2x) = R(q^2x,\, x,\, q^2x,\, x). \tag{19}$$

Equalities (6), (15), (16) and (19) imply

$$\mu(L_1; p) = \mu(L_2; p) = \mu(L_3; p) = \frac{R(x,\, qx,\, x,\, qx)}{g^2(x,x) - g^2(x,qx)}\,.$$

By virtue of the linear independence of x and qx, we have

$$g^2(x,x) - g^2(x,qx) = g^2(x,x)(1 - \cos\varphi) \neq 0,$$

where φ is the angle between x and qx.

$\square$

5. An Orthonormal q-Base of Vectors in T_pM

Let M be the Riemannian manifold with a metric tensor g and a structure q, defined by (1) and (3), respectively. We note that the only real eigenvalue and the only eigenvector of the structure q are $\lambda = 1$ and $x\big(x^1,\, x^1,\, x^1\big)$, respectively.

Now, let

$$x = \big(x^1,\, x^2,\, x^3\big) \tag{20}$$

be a non-eigenvector vector of the structure q. We have

$$g(x,x) = \|x\|\|x\|\cos 0 = \|x\|^2, \quad g(x,qx) = \|x\|\|qx\|\cos\varphi = \|x\|^2\cos\varphi, \tag{21}$$

where $\|x\|$ and $\|qx\|$ are the norms of x and qx; and φ is the angle between x and qx.

From (1), (20) and (21) we calculate

$$g(x,x) = A\left((x^1)^2 + (x^2)^2 + (x^3)^2\right) + 2B\left(x^1x^2 + x^1x^3 + x^2x^3\right), \tag{22}$$

$$g(x,qx) = B\left((x^1)^2 + (x^2)^2 + (x^3)^2\right) + (A+B)\left(x^1x^2 + x^1x^3 + x^2x^3\right). \tag{23}$$

The above equations imply $\|x\| = \|qx\| > 0$.

Theorem 4. *Let M be the Riemannian manifold with a metric tensor g and an affinor structure q, defined by (1) and (3), respectively. Let $x(x^1, x^2, x^3)$ be a non-eigenvector on T_pM. If φ is the angle between x and qx, then we have $\varphi \in \left(0, \dfrac{2\pi}{3}\right)$.*

Proof. We apply equations (22) and (23) in $\cos\varphi = \dfrac{g(x,qx)}{g(x,x)}$, and we get

$$\cos\varphi = \frac{\big((x^1)^2 + (x^2)^2 + (x^3)^2\big) + (A+B)\big(x^1x^2 + x^1x^3 + x^2x^3\big)}{A\big((x^1)^2 + (x^2)^2 + (x^3)^2\big) + 2B\big(x^1x^2 + x^1x^3 + x^2x^3\big)}. \tag{24}$$

Also we have $x(x^1,\, x^2,\, x^3) \neq (x^1,\, x^1,\, x^1)$ because x is a non-eigenvector of q.

We suppose that $\varphi \geq \dfrac{2\pi}{3}$, i.e. $\cos\varphi \leq -\dfrac{1}{2}$. The last condition and (24) imply

$$\frac{B\big((x^1)^2 + (x^2)^2 + (x^3)^2\big) + (A+B)\big(x^1x^2 + x^1x^3 + x^2x^3\big)}{A\big((x^1)^2 + (x^2)^2 + (x^3)^2\big) + 2B\big(x^1x^2 + x^1x^3 + x^2x^3\big)} \leq -\frac{1}{2}$$

that gives the inequality

$$(2B+A)\Big((x^1)^2+(x^2)^2+(x^3)^2+2\big(x^1x^2+x^1x^3+x^2x^3\big)\Big)\leq 0.$$

From the condition $A+2B>0$ we get that

$$(x^1)^2+(x^2)^2+(x^3)^2+2\big(x^1x^2+x^1x^3+x^2x^3\big)\leq 0$$

and $(x^1+x^2+x^3)^2\leq 0$. The last inequality has no solution in the real set. Then we have $\cos\varphi>-\dfrac{1}{2}$.

□

Immediately, from Theorem 4, we establish that an orthonormal q-base $(x,\, qx,\, q^2x)$ in T_pM exists. Particularly, we verify that the vector

$$x=\left(\frac{\sqrt{A-B}+\sqrt{A+3B}}{2\sqrt{A^2+AB-2B^2}}\,,\quad \frac{\sqrt{A-B}-\sqrt{A+3B}}{2\sqrt{A^2+AB-2B^2}}\,,\quad 0\right) \tag{25}$$

satisfies the conditions

$$g(x,x)=1,\qquad g(x,qx)=0. \tag{26}$$

The base $(x,\, qx,\, q^2x)$, where x satisfies (25), is an example of an orthonormal q-base in T_pM.

Theorem 5 *Let M be the Riemannian manifold with a metric tensor g and a parallel structure q, defined by (1) and (3), respectively. Let $(x,\, qx,\, q^2x)$ be an orthonormal q-base in T_pM, $p\in M$, and $u=\alpha\,.\,x+\beta\,.\,qx+\gamma\,.\,q^2x$, $v=\delta.\,x+\zeta.\,qx+\eta.\,q^2x$ be arbitrary vectors in T_pM. For the sectional curvature $\mu(u,v)$ of 2-plane $\{u,v\}$ we have*

$$\mu(u,v)=\frac{\big(\alpha\zeta-\beta\delta+\delta\gamma-\alpha\eta+\beta\eta-\gamma\zeta\big)^2}{\big(\alpha^2+\beta^2+\gamma^2\big)\big(\delta^2+\zeta^2+\eta^2\big)-\big(\alpha\delta+\beta\zeta+\gamma\eta\big)^2}\,\mu(x,qx). \tag{27}$$

Proof. We calculate

$$g(u,u)=\alpha^2+\beta^2+\gamma^2,\quad g(v,v)=\delta^2+\zeta^2+\eta^2, \tag{28}$$
$$g(u,v)=\alpha\delta+\beta\zeta+\gamma\eta.$$

For the sectional curvature of 2-plane $\{u,v\}$ we have

$$\mu(u,v)=\frac{R(u,\,v,\,u,\,v)}{g(u,u)g(v,v)-g^2(u,v)}\,. \tag{29}$$

Using the linear properties of the metric g and the curvature tensor field R after long calculations we get

$$
\begin{aligned}
R(u,\, v,\, u,\, v) =&(\alpha\zeta-\beta\delta)^2R(x,\, qx,\, x,\, qx)\\
&+(\delta\gamma-\alpha\eta)^2R(x,\, q^2x,\, x,\, q^2x)\\
&+(\beta\eta-\gamma\zeta)^2R(qx,\, q^2x,\, qx,\, q^2x)\\
&+2(\alpha\zeta-\beta\delta)(\delta\gamma-\alpha\eta)R(x,\, qx,\, q^2x,\, x)\\
&+2(\delta\gamma-\alpha\eta)(\beta\eta-\gamma\zeta)R(q^2x,\, x,\, qx,\, q^2x)\\
&+2(\alpha\zeta-\beta\delta)(\beta\eta-\gamma\zeta)R(x,\, qx,\, qx,\, q^2x).
\end{aligned}
\tag{30}
$$

From (18), (19) and (30) we obtain

$$
R(u,\, v,\, u,\, v)=\big((\alpha\zeta-\beta\delta)+(\delta\gamma-\alpha\eta)+(\beta\eta-\gamma\zeta)\big)^2R(x,\, qx,\, x,\, qx). \tag{31}
$$

From (28), (29) and (31) we get

$$
\mu(u,v)=\frac{\big(\alpha\zeta-\beta\delta+\delta\gamma-\alpha\eta+\beta\eta-\gamma\zeta\big)^2}{(\alpha^2+\beta^2+\gamma^2)(\delta^2+\zeta^2+\eta^2)-(\alpha\delta+\beta\zeta+\gamma\eta)^2}R(x,\, qx,\, x,\, qx).
$$

The last equation and (26) imply (27).

□

Corollary 1. *Let u be an arbitrary non-eigenvector in T_pM, $p\in M$, and θ be the angle between u and qu.*
Then we have

$$
\mu(u,qu)=\mu(x,qx)\tan^2\frac{\theta}{2}\,,\quad \theta\in\left(0,\frac{2\pi}{3}\right). \tag{32}
$$

Proof. In (27) we substitute $v=qu$, $\delta=\gamma$, $\zeta=\alpha$, $\eta=\beta$ and we obtain

$$
\mu(u,qu)=\frac{(\alpha^2+\beta^2+\gamma^2-\beta\gamma-\alpha\beta-\alpha\gamma)^2}{(\alpha^2+\beta^2+\gamma^2)^2-(\alpha\gamma+\alpha\beta+\gamma\beta)^2}\,\mu(x,qx).
$$

Then from (28) we get

$$
\mu(u,qu)=\frac{(g(u,u)-g(u,qu))^2}{g^2(u,u)-g^2(u,qu)}\,\mu(x,qx),
$$

i.e.

$$
\mu(u,qu)=\frac{(1-\cos\theta)^2}{1-\cos^2\theta}\,\mu(x,qx),
$$

which implies (32).

□

Corollary 2. *Let u, v be an arbitrary non-eigenvectors on T_pM, $p \in M$, and θ be the angle between u and qu, and ψ be the angle between v and qv. Then we have*

$$\mu(u,qu)\tan^2\frac{\psi}{2} = \mu(v,qv)\tan^2\frac{\theta}{2}, \quad \psi,\, \theta \in \left(0, \frac{2\pi}{3}\right).$$

The proof follows immediately from (32).

Acknowledgments

Research was partially supported by project RS11 - FMI - 004 of the Scientific Research Fund, Paisii Hilendarski University of Plovdiv, Bulgaria.

References

[1] Borisov A., Ganchev G., Curvature properties of Kaehlerian manifolds with B-metric, *Math. Educ. Math., Proc of 14th Spring Conf. of UBM*, Sunny Beach, (1985), 220–226.

[2] Grey A., Curvature identities for Hermitian and Almost Hermitian Manifolds, *Tohoku Math. Journal*, Vol. 28, No. 4, (1976), 601–612.

[3] Gribachev K., Mekerov D., Djelepov G., On the Geometry of Almost B-manifolds, *Compt. Rend. Acad. Bulg. Sci.*, Vol. 38, No. 5, (1985), 563–566.

[4] Gribachev K., Djelepov G., On the Geometry of the normal generalized B-manifolds, *PU Sci. Works-math.*, Vol. 23, No. 1, (1985), 157–168.

[5] Manev M., Nakova G., Curvature properties on some three-dimensional almost contact B-metric manifolds, *Plovdiv Univ. Sci. Works - math.*, Vol. 34, no. 3, (2004), 51-60.

[6] Nakova G., Manev M., Curvature properties on some three-dimensional almost contact manifolds with B-metric,*Proc. 5th Int. Conf. Geometry, Integrability & Quantization V Eds.I. M. Mladenov, A. C. Hirshfeld*, SOFTEX, Sofia, (2004), 169-177.

[7] Yano K., Ishihara S., Structure defined by f $f^3 + f = 0$, *Proc.US-Japan Seminar in Differential Geometry.*, Kyoto, (1965), 153–166.

[8] Yano K., On a structure defined by a tensor field of type $(1,1)$ satisfying $f^3 + f = 0$, Tensor, (1963), 99–109.

Georgi Dzhelepov, Dimitar Razpopov
Department of Mathematics and Physics
Agricultural University of Plovdiv
12 Mendeleev Blvd., 4000 Plovdiv, Bulgaria
e-mail: drazpopov@qustyle.bg

Received 05 December 2011

Iva Dokuzova
Faculty of Mathematics and Informatics
University of Plovdiv
236 Bulgaria Blvd., 4003 Plovdiv, Bulgaria
e-mail: dokuzova@uni-plovdiv.bg

ВЪРХУ ТРИМЕРНО РИМАНОВО МНОГООБРАЗИЕ С ДОПЪЛНИТЕЛНА СТРУКТУРА

Георги Джелепов, Ива Докузова, Димитър Разпопов

Резюме. Много научни работи са посветени на диференциалните многообразия, допускащи допълнителна афинорна структура f. Повечето от тях разглеждат структура на почти произведение или почти комплексна структура. Ще отбележим, че К. Яно има публикации, в които структурата удовлетворява уравнение за третата си степен.

Ако многообразието допуска афинна свързаност ∇, то при определени условия за ∇f се получава тъждество за кривинния тензор. Такава задача е решена в почти ермитовата геометрия от А. Грей, а по-късно за почти комплексните многообразия с норденова метрика от А. Борисов, Г. Ганчев, К. Грибачев, Г. Джелепов, Д. Мекеров.

В настоящата работа разглеждаме тримерно риманово многообразие с допълнителна афинорна структура, чиято трета степен е идентитетът. Локалните координати на метричната структура и на допълнителната структура са циркулантни. Намираме условия, при които структурата е паралелна по отношение на римановата свързаност. В този случай получаваме някои тъждества за кривините.