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FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH TWO CIRCULANT STRUCTURES*

Dimitar Razpopov

We consider a class (M, g, q) of four-dimensional Riemannian manifolds M, where beside the metric g there is an additional structure q, whose fourth power is the unit matrix. We use the existence of a local coordinate system for which coordinates of g and q are circulant matrices. In this system q has constant coordinates and q is an isometry with respect to g. By the special identity for the curvature tensor R generated by the connection ∇ of g we define a subclass of (M, g, q). For any (M, g, q) in this subclass we get some assertions for the sectional curvatures of two-planes. We get the necessary and sufficient condition for g such that q is parallel with respect to ∇.

1. Introduction. The main purpose of the present paper is to continue the considerations on some Riemannian manifolds using the existence of an useful local circulant coordinate system analogously to [3], [4], [5].

In Section 2 we introduce four-dimensional differentiable manifold M with a Riemannian metric g whose matrix in local coordinates is a special circulant matrix. Furthermore, we consider an additional structure q on M with $q^4 = \mathrm{id}$ such that its matrix in local coordinates is also circulant. Thus, the structure q is an isometry with respect to g. We denote by (M, g, q) the manifold M equipped with the metric g and the structure q. In Section 3 in Theorem 3.4 we obtain that an orthogonal basis of type $\{x, qx, q^2x, q^3x\}$ exists in the tangent space of a manifold (M, g, q). In Section 4 we establish relations between the sectional curvatures of some special 2-planes in the tangent space. In Section 5 we obtain a necessary and sufficient condition for q to be parallel with respect to the Riemannian connection of g.

2. Preliminaries. Let M be a four-dimensional manifold with a Riemannian metric g. Let the local components of the metric g at an arbitrary point $p(X^1, X^2, X^3, X^4) \in M$ form the following circulant matrix:

$$(g_{ij}) = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}, \tag{1}$$

where $A = A(p), B = B(p), C = C(p)$ are smooth functions.

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We suppose

$$0 < B < C < A \,. \tag{2}$$

Then the conditions to be a positive definite metric g are satisfied:

$$A > 0, \quad \begin{vmatrix} A & B \\ B & A \end{vmatrix} = (A-B)(A+B) > 0,$$

$$\begin{vmatrix} A & B & C \\ B & A & B \\ C & B & A \end{vmatrix} = (A-C)\Big(A(C+A) - 2B^2\Big) > 0,$$

$$\begin{vmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{vmatrix} = (A-C)^2\Big((A+C)^2 - 4B^2\Big) > 0.$$

We denote by (M, g) the manifold M equipped with the Riemannian metric g defined by (1) with conditions (2).

Let q be an endomorphism in the tangent space T_pM of the manifold (M, g). We suppose the local coordinates of q are given by the circulant matrix

$$(q_i^s) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Then q satisfies

$$q^4 = \mathrm{id}, \qquad q^2 \neq \pm\mathrm{id}. \tag{4}$$

We denote by (M, g, q) the manifold (M, g) equipped with the structure q, defined by (3).

Further, x, y, z, u will stand for arbitrary elements of the algebra on the smooth vector fields on M or vectors in the tangent space T_pM. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3, 4\}$.

From (1) and (3) we get immediately the following

Theorem 2.1. *The structure q of the manifold (M, g, q) is an isometry with respect to the metric g, i.e.*

$$g(qx, qy) = g(x, y). \tag{5}$$

3. Orthogonal q-bases of T_pM.

Definition 3.1. A basis of type $\{x, qx, q^2x, q^3x\}$ of T_pM is called a *q-basis*. In this case we say that *the vector x induces a q-basis of T_pM*.

Obviously, we have the following

Proposition 3.2. *A vector $x = (x^1, x^2, x^3, x^4)$ induces a q-basis of T_pM if and only if*

$$\Big((x^1 - x^3)^2 + (x^2 - x^4)^2\Big)\Big((x^1 + x^3)^2 - (x^2 + x^4)^2\Big) \neq 0 \tag{6}$$

Proof. If $x = (x^1, x^2, x^3, x^4) \in T_pM$, then $qx = (x^2, x^3, x^4, x^1)$, $q^2x = (x^3, x^4, x^1, x^2)$, $q^3x = (x^4, x^1, x^2, x^3)$. The determinant of coordinates of the vectors x, qx, q^2x, q^3x is just the left side of (6). The vectors x, qx, q^2x, q^3x are linearly independent which imply (6). $\square$

Theorem 3.3. *If $x = (x^1, x^2, x^3, x^4)$ induces a q-basis of T_pM, then for the angles $\measuredangle(x, qx)$, $\measuredangle(x, q^2x)$, $\measuredangle(qx, q^2x)$, $\measuredangle(qx, q^3x)$, $\measuredangle(x, q^3x)$ and $\measuredangle(q^2x, q^3x)$ we have*

$$\measuredangle(x, qx) = \measuredangle(qx, q^2x) = \measuredangle(x, q^3x) = \measuredangle(q^2x, q^3x), \quad \measuredangle(x, q^2x) = \measuredangle(qx, q^3x).$$

Proof. Evidently from (5) we have $g(q^3x, q^3y) = g(q^2x, q^2y) = g(qx, qy) = g(x, y)$. Then from the well known formula

$$\cos \measuredangle(x, y) = \frac{g(x, y)}{\sqrt{g(x, x)}\sqrt{g(y, y)}}$$

we get $\cos \measuredangle(x, qx) = \cos \measuredangle(qx, q^2x) = \cos \measuredangle(x, q^3x) = \cos \measuredangle(q^2x, q^3x)$ and $\cos \measuredangle(x, q^2x) = \cos \measuredangle(qx, q^3x)$. $\square$

Theorem 3.4. *Let x induce a q-basis in T_pM of a manifold (M, g, q). Then there exists an orthogonal q-basis $\{x, qx, q^2x, q^3x\}$ in T_pM.*

Proof. Let $\{x, qx, q^2x, q^3x\}$ be a q-basis in T_pM of a manifold (M, g, q). Then the triples of vectors $\{x, qx, q^2x\}$; $\{x, qx, q^3x\}$; $\{x, q^2x, q^3x\}$; $\{qx, q^2x, q^3x\}$ form four congruent pyramids. We consider for example one of them formed by $\{x, qx, q^2x\}$. Its first face is isosceles triangle with angles $\measuredangle(x, qx) = \varphi$, $\frac{\pi - \varphi}{2}$, $\frac{\pi - \varphi}{2}$. Its second face is isosceles triangle with angles $\measuredangle(qx, q^2x) = \varphi$, $\frac{\pi - \varphi}{2}$, $\frac{\pi - \varphi}{2}$. Its third face is isosceles triangle with angles $\measuredangle(x, q^2x) = \theta$, $\frac{\pi - \theta}{2}$, $\frac{\pi - \theta}{2}$. The fourth face is isosceles triangle with angles $\measuredangle(x - qx, q^2x - qx) = \phi$, $\frac{\pi - \phi}{2}$ and $\frac{\pi - \phi}{2}$. From the Cosine Rule applied to the fourth side and from (5) we get

$$2g(x, x)(1 - \cos\theta) = 4g(x, x)(1 - \cos\varphi)\cos\phi,$$

and then

$$\cos\phi = \frac{1 - 2\cos\varphi + cos\theta}{2(1 - cos\varphi)}.$$

From the above and $-1 < \cos\phi < 1$ we find

$$4\cos\varphi - \cos\theta < 3.$$

The angles $\varphi = \frac{\pi}{2}$, $\theta = \frac{\pi}{2}$ satisfy the above inequality. Having in mind Theorem 3.3 we prove that there exists an orthogonal q-basis in T_pM. $\square$

4. Curvature properties of (M, g, q). Let ∇ be the Riemannian connection of g for a manifold (M, g, q). Let R be the curvature tensor field of ∇ of type $(0, 4)$, and R satisfies the identity

$$R(x, y, qz, qu) = R(x, y, z, u). \tag{7}$$

We note, that by identities like (7) in [1], [2] the subclass of almost complex manifolds with Norden metric and the subclass of almost Hermitian manifolds respectively have been defined.

The sectional curvature μ of 2-plane $\{x, y\}$ from T_pM is expressed by the formula [6]

$$\mu(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g^2(x, y)}. \tag{8}$$

Theorem 4.1. *Let (M, g, q) be a manifold with property (7). Let x induce a q-basis* *in T_pM. Then for the sectional curvature μ of 2-planes we have*

$$\mu(x, qx) = \mu(qx, q^2x) = \mu(q^2x, q^3x) = \mu(q^3x, x), \tag{9}$$

$$\mu(x, q^2x) = \mu(qx, q^3x) = 0. \tag{10}$$

Proof. From (7) we find

$$R(x, y, z, u) = R(x, y, qz, qu) = R(x, y, q^2z, q^2u) = R(x, y, q^3z, q^3u). \tag{11}$$

In (11) we substitute

1) u for qx, y for qx and z for x;
2) z for x, y for q^2x and u for q^2x;
3) z for x, y for q^3x and u for q^3x

and obtain respectively

$$R(x, qx, x, qx) = R(x, qx, qx, q^2x) = R(x, qx, q^2x, q^3x) = R(x, qx, q^3x, qx), \tag{12}$$

$$R(x, q^2x, x, q^2x) = R(x, q^2x, qx, q^3x) = R(x, q^2x, q^2x, x) = R(x, q^2x, q^3x, x), \tag{13}$$

$$R(x, q^3x, x, q^3x) = R(x, q^3x, qx, x) = R(x, q^3x, q^2x, x) = R(x, q^3x, q^3x, q^2x). \tag{14}$$

Using (12), (14) and (8) we get (9) and using (13) and (8) we get (10). □

We see that every 2-plane $\{x, qx\} \in T_pM$ has only two q-bases $\{x, qx\}$ or $\{-x, -qx\}$. So the sectional curvature μ of $\{x, qx\}$ is a function of the $\measuredangle(x, qx) = \varphi$, i.e. $\mu(x, qx) = \mu(\varphi)$.

Proposition 4.2. *Let (M, g, q) be a manifold with property (7) and u induce a q-basis in T_pM. If $\{x, qx, q^2x, q^3x\}$ is an orthonormal q-basis in T_pM, then the sectional curvature satisfies*

$$\mu(\varphi) = \frac{1}{1 - \cos^2\varphi}\mu(\frac{\pi}{2}), \tag{15}$$

where $\varphi = \measuredangle(u, qu)$.

Proof. Let $u = \alpha x + \beta qx + \gamma q^2x + \delta q^3x$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Then $qu = \delta x + \alpha qx + \beta q^2x + \gamma q^3x$, $q^2u = \gamma x + \delta qx + \alpha q^2x + \beta q^3x$ and $q^3u = \beta x + \gamma qx + \delta q^2x + \alpha q^3x$. We calculate

$$\cos\varphi = \alpha\beta + \alpha\delta + \beta\gamma + \delta\gamma; \qquad \cos\theta = 2\alpha\gamma + 2\beta\delta, \tag{16}$$

where $\theta = \measuredangle(u, q^2u)$. Then using the linear properties of the curvature tensor R and having in mind (12)–(14), we obtain

$$\begin{aligned} R(u, qu, u, qu) = \Big(&(\alpha^2 + \gamma^2 - 2\beta\delta)^2 + (\beta^2 + \delta^2 - 2\gamma\alpha)^2 \\ &+ 2(\alpha^2 + \gamma^2 - 2\beta\delta)(\beta^2 + \delta^2 - 2\gamma\alpha)\Big)R(x, qx, x, qx). \end{aligned} \tag{17}$$

From (16) we get

$$\begin{aligned} (1 - \cos\theta)^2 R(u, qu, u, qu) = \Big(&(\alpha^2 + \gamma^2 - 2\beta\delta)^2 + (\beta^2 + \delta^2 - 2\gamma\alpha)^2 \\ &+ 2(\alpha^2 + \gamma^2 - 2\beta\delta)(\beta^2 + \delta^2 - 2\gamma\alpha)\Big)R(x, qx, x, qx) \end{aligned} \tag{18}$$

We substitute (17) and (18) in (8) and obtain (15). □

5. Parallelity of the circulant structure q.

Theorem 5.1. *Let ∇ be the Riemannian connection of g of a manifold (M, g, q). Then the structure q is parallel with respect to the Riemannian connection ∇ if and only* *if*

$$\text{grad}A = (\text{grad}C)q^2\ , \quad 2\text{grad}B = (\text{grad}C)(q + q^3), \tag{19}$$

where grad$A$, grad$B$ *and* grad$C$ *are gradients of the functions* A, B *and* C.

Proof. Let the structure q be parallel with respect to the Riemannian connection ∇ of a manifold (M, g, q), i.e. $\nabla q = 0$. Let Γ^s_{ij} be the Christoffel symbols of ∇. If $\nabla q = 0$, then

$$\nabla_i q^s_j = \partial_i q^s_j + \Gamma^s_{ik} q^k_j - \Gamma^k_{ij} q^s_k = 0. \tag{20}$$

From (3) and (20) we get

$$\Gamma^s_{ik} q^k_j = \Gamma^k_{ij} q^s_k. \tag{21}$$

We denote

$$A_i = \frac{\partial A}{\partial X^i}\ , \quad B_i = \frac{\partial B}{\partial X^i}\ , \quad C_i = \frac{\partial C}{\partial X^i}\ , \tag{22}$$

where A, B and C are the functions from (1).

We find the inverse matrix of (g_{ij}) as follows:

$$(g^{ij}) = \frac{1}{D}\begin{pmatrix} \overline{A} & \overline{B} & \overline{C} & \overline{B} \\ \overline{B} & \overline{A} & \overline{B} & \overline{C} \\ \overline{C} & \overline{B} & \overline{A} & \overline{B} \\ \overline{B} & \overline{C} & \overline{B} & \overline{A} \end{pmatrix}, \qquad D = (A - C)((A + C)^2 - 4B^2), \tag{23}$$

where $\overline{A} = A(A + C) - 2B^2$, $\overline{B} = B(C - A)$, $\overline{C} = 2B^2 - C(A + C)$.

Using (1), (3), (21)–(23) and the well known identities

$$2\Gamma^s_{ij} = g^{as}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}), \tag{24}$$

after a long computation we get the following system:

$$\begin{aligned}
&A_4 - B_1 + B_3 - C_2 = 0,\\
&A_4 + B_1 - B_3 - C_2 = 0,\\
&2A_2 + A_4 - 3B_1 - B_3 + C_2 = 0,\\
&A_3 + B_2 - B_4 - C_1 = 0,\\
&A_3 - B_2 + B_4 - C_1 = 0,\\
&A_2 - B_1 + B_3 - C_4 = 0,\\
&A_2 + B_1 - B_3 - C_4 = 0,\\
&A_4 - B_1 + 3B_3 + C_2 + 2C_4 = 0,\\
&A_2 + 2A_4 - 3B_1 - B_3 + C_4 = 0,\\
&A_2 + 2A_4 - B_1 - 3B_3 + C_4 = 0,\\
&A_1 + 2A_3 - 3B_2 - B_4 + C_3 = 0,\\
&A_1 - B_2 + B_4 - C_3 = 0,\\
&A_3 - B_2 - 3B_4 + C_1 + 2C_3 = 0,\\
&A_1 - B_2 - 3B_4 + 2C_1 + C_3 = 0,\\
&2A_1 + A_3 - B_2 - 3B_4 + C_1 = 0,\\
&A_2 - B_1 - 3B_3 + 2C_2 + C_4 = 0.
\end{aligned}$$

The last system implies

$$
\begin{aligned}
&A_1 = C_3,\ A_2 = C_4,\ A_3 = C_1,\ A_4 = C_2,\ B_1 = B_3, \\
&B_2 = B_4,\ 2B_1 = C_4 + C_2,\ 2B_2 = C_1 + C_3.
\end{aligned} \tag{25}
$$

Then we obtain that (19) is valid.

Inversely, let (19) be valid. We can verify that (25) is valid, too. The identities (25) imply (21) and consequently (20) is true. So $\nabla q = 0$. □

Proposition 5.2. *Let (M, g, q) be a manifold with parallel structure q with respect of g. Then (M, g, q) is a manifold with property* (7).

Proof. The condition $\nabla q = 0$ implies $\nabla_i q_s^j = 0$. The integrability condition of this system is

$$R^a_{jkl} q^s_a = R^s_{akl} q^a_j, \tag{26}$$

where R^a_{jkl} are the local coordinates of R. From (26) we find

$$R_{ajkl} q^a_{.s} = R_{sakl} q^a_{j.}. \tag{27}$$

We get $q^a_{.s}$ are the local coordinates of q^3. So (27) implies

$$R(q^3 u, v, w, t) = R(u, qv, w, t)$$

from which (7) follows. Then (M, g, q) has the property (7). □

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Dimitar Razpopov
Department of Mathematics and Physics
Agricultural University of Plovdiv
4000 Plovdiv, Bulgaria
e-mail: dimitrerazpopov@hotmail.com

ЧЕТИРИМЕРНИ РИМАНОВИ МНОГООБРАЗИЯ С ДВЕ ЦИРКУЛАНТНИ СТРУКТУРИ

Димитър Руменов Разпопов

Разглеждаме един клас (M, g, q) на четиримерни риманови многообразия M, където освен с метрика g многообразието е снабдено с допълнителна структура q, чиято четвърта степен е идентитетът. Използваме съществуването на локална координатна система, в която координатите на g и q са циркулантни матрици. В тази координатна система координатите на q са константи и q е изометрия по отношение на g. Чрез специално тъждество за тензора на кривина R породен от свързаността ∇ на g дефинираме един подклас (M, g, q). За всяко (M, g, q) от този подклас получаваме твърдения за секционните кривини на двумерни q-площадки. Намираме необходимо и достатъчно условие за g, така че q да е паралелна по отношение на ∇.