J. Geom. (2020) 111:9

0047-2468/20/010001-18
published online January 23, 2020
https://doi.org/10.1007/s00022-020-0521-z

Journal of Geometry

Four-dimensional almost Einstein manifolds with skew-circulant stuctures

Iva Dokuzova and Dimitar Razpopov

Abstract. We consider a four-dimensional Riemannian manifold M with an additional structure S, whose fourth power is minus identity. In a local coordinate system the components of the metric g and the structure S form skew-circulant matrices. Both structures S and g are compatible, such that an isometry is induced in every tangent space of M. By a special identity for the curvature tensor, generated by the Riemannian connection of g, we determine classes of Einstein and almost Einstein manifolds. For such manifolds we obtain propositions for the sectional curvatures of some characteristic 2-planes in a tangent space of M. We consider a Hermitian manifold associated with the studied manifold and find conditions for g, under which it is a Kähler manifold. We construct some examples of the considered manifolds on Lie groups.

Mathematics Subject Classification. 53B20, 53C15, 53C25, 53C55.

Keywords. Riemannian manifold, Einstein manifold, sectional curvatures, Ricci curvature, Lie group.

1. Introduction

The right circulant matrices and the right skew-circulant matrices are Toeplitz matrices, which are well-studied in [1,3]. The set of invertible circulant (skew-circulant) matrices form a group with respect to the matrix multiplication. Such matrices have application to geometry, linear codes, graph theory, vibration analysis (for example [2,7,9,11,13,14]).

A. Gray, L. Hervella and L. Vanhecke used curvature identities to classify and to study the almost Hermitian manifolds (for instance in [4–6,15]). The Hermitian manifolds form a class of manifolds with an integrable almost complex structure J. The class of the Kähler manifolds is their subclass and such

This work is partially supported by Project MU19-FMI-020 of the Scientific Research Fund, Paisii Hilendarski University of Plovdiv, Bulgaria.

manifolds have a parallel structure J. According to A. Gray, the Kähler manifolds have an especially rich geometric structure, due to the Kähler curvature identity $R(\cdot,\cdot,J\cdot,J\cdot)=R(\cdot,\cdot,\cdot,\cdot)$. Some of the recent investigations on the curvature properties of the almost Hermitian manifolds are made in [8,10,12,16].

In the present work we study a four-dimensional differentiable manifold M with a Riemannian metric g. The manifold M is equipped locally with an additional structure S, which satisfies $S^4=-\mathrm{id}$. The component matrix of S is a special skew-circulant matrix, i.e., S is a skew-circulant structure. Moreover, S is compatible with g, such that an isometry is induced in every tangent space of M. Such a manifold (M,g,S) is associated with a Hermitian manifold (M,g,J), where $J=S^2$ is a complex structure.

The paper is organized as follows. In Sect. 2, we introduce a manifold (M,g,S) and give some necessary facts for our investigations. In Sect. 3, we obtain a class of almost Einstein manifolds (M,g,S) and a class of Einstein manifolds (M,g,S). In Sect. 4, we get conditions under which an orthogonal basis of type $\{S^3x,S^2x,Sx,x\}$ exists in every tangent space of (M,g,S). In Sect. 5, we find some curvature properties of the considered Einstein and almost Einstein manifolds. In Sect. 6, we obtain a necessary and sufficient condition for S to be parallel with respect to the Riemannian connection of g. Also, we get conditions for (M,g,J) to be a Kähler manifold. In Sect. 7, we construct examples of the considered manifolds on Lie groups and find some of their geometric characteristics.

2. Preliminaries

Let M be a 4-dimensional Riemannian manifold equipped with an endomorphism S in every tangent space T_pM at a point p on M. Let the coordinates of S, with respect to some basis $\{e_i\}$, form a right skew-circulant matrix as follows

$$(S_j^k)=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-1&0&0&0\end{pmatrix}. \tag{2.1}$$

We use local coordinates to facilitate our calculations.

According to (2.1) S has the property

$$S^4=-\mathrm{id}. \tag{2.2}$$

We assume that the metric g and the structure S satisfy

$$g(Sx,Sy)=g(x,y). \tag{2.3}$$

Here and anywhere in this work, x,y,z,u will stand for arbitrary elements of the algebra on smooth vector fields on M or vectors in T_pM. The Einstein summation convention is used, the range of the summation indices being always $\{1,2,3,4\}$.

The conditions (2.1) and (2.3) imply that the matrix of g, with respect to the local basis $\{e_i\}$, has the form

$$(g_{ij}) = \begin{pmatrix} A & B & 0 & -B \\ B & A & B & 0 \\ 0 & B & A & B \\ -B & 0 & B & A \end{pmatrix}, \tag{2.4}$$

i.e., it is right skew-circulant. Here $A = A(p)$ and $B = B(p)$ are smooth functions of an arbitrary point $p(X^1, X^2, X^3, X^4)$ on M. The determinant of the matrix (2.4) has the value $\det(g_{ij}) = (A^2 - 2B^2)^2$. It is supposed that

$$A(p) > \sqrt{2}B(p) > 0 \tag{2.5}$$

in order g to be positive definite.

A manifold M introduced in this way we denote by (M, g, S).

Now, we consider an associated metric $\tilde{g}$ with g, determined by

$$\tilde{g}(x, y) = g(x, Sy) + g(Sx, y). \tag{2.6}$$

Using (2.1), (2.4) and (2.6) we get that the matrix of its components is

$$(\tilde{g}_{ij}) = \begin{pmatrix} 2B & A & 0 & -A \\ A & 2B & A & 0 \\ 0 & A & 2B & A \\ -A & 0 & A & 2B \end{pmatrix}. \tag{2.7}$$

Two of the eigenvalues of (2.7) are $2B - \sqrt{2}A$ and the other two are $2B + \sqrt{2}A$. Since inequalities (2.5) are valid, $\tilde{g}$ has signature $(2, 2)$. So $\tilde{g}$ is an indefinite metric.

The inverse matrices of (g_{ij}) and $(\tilde{g}_{ij})$ are as follows:

$$(g^{ij}) = \frac{1}{A^2 - 2B^2} \begin{pmatrix} A & -B & 0 & B \\ -B & A & -B & 0 \\ 0 & -B & A & -B \\ B & 0 & -B & A \end{pmatrix}, \tag{2.8}$$

$$(\tilde{g}^{ij}) = \frac{1}{2(A^2 - 2B^2)} \begin{pmatrix} -2B & A & 0 & -A \\ A & -2B & A & 0 \\ 0 & A & -2B & A \\ -A & 0 & A & -2B \end{pmatrix}. \tag{2.9}$$

Let ∇ be the Riemannian connection of g. The curvature tensor R of ∇ is determined by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z. \tag{2.10}$$

The tensor of type $(0, 4)$ associated with R is defined by

$$R(x, y, z, u) = g(R(x, y)z, u). \tag{2.11}$$

The Ricci tensor ρ with respect to g is given by the well-known formula

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j). \tag{2.12}$$

The scalar curvature τ with respect to g and its associated quantity τ^* are determined by

$$\tau = g^{ij}\rho(e_i, e_j), \qquad \tau^* = \tilde{g}^{ij}\rho(e_i, e_j). \tag{2.13}$$

Now, we consider a manifold (M, g, S) with the condition

$$\nabla S = 0. \tag{2.14}$$

i.e., S is a parallel structure with respect to ∇.

Proposition 2.1. *Every manifold (M, g, S) with a parallel structure S satisfies the curvature identity*

$$R(x, y, Sz, Su) = R(x, y, z, u). \tag{2.15}$$

Proof. The well-known formula $(\nabla_x S)y = \nabla_x Sy - S\nabla_x y$, together with (2.14), yields

$$\nabla_x Sy = S\nabla_x y. \tag{2.16}$$

On the other hand, the equality (2.10) implies

$$R(x, y, Sz, Su) = g\big(R(x, y)Sz, Su\big).$$

Because of the latter identity, using (2.3), (2.10), (2.11) and (2.16), we get (2.15). □

Due to the last proposition, we note that the identity (2.15) defines a more general class of manifolds (M, g, S) than the class with the condition (2.14). Farther in this paper, we will investigate the properties of manifolds in these two classes.

3. Almost Einstein manifolds

In this section we consider manifolds (M, g, S) with the property (2.15).

By R_{ijkh} and ρ_{ij} we will denote the components of the curvature tensor R and the components of the Ricci tensor ρ with respect to the local basis $\{e_i\}$, respectively. Hence, we establish the following propositions.

Proposition 3.1. *The property (2.15) of the curvature tensor R of (M, g, S) is equivalent to the conditions*

$$\begin{aligned} R_{1313} =& R_{2424} = R_{1324} = 2R_{1212} = 2R_{1414} = 2R_{2323} = 2R_{3434} \\ &= 2R_{1223} = 2R_{1214} = 2R_{1434} = 2R_{1234} = 2R_{2334} = 2R_{2314}, \\ R_{1213} =& R_{1224} = R_{1413} = R_{2414} = R_{2423} = R_{2313} = R_{1334} = R_{2434}. \end{aligned} \tag{3.1}$$

Proof. The local form of (2.15) is

$$R_{ijlm}S_k^l S_h^m = R_{ijkh}. \tag{3.2}$$

Then, using (2.1), we find the equalities

$$\begin{aligned} R_{1313} =& R_{2424} = R_{1324}, \\ R_{1212} =& R_{1414} = R_{2323} = R_{3434} = R_{1223} = R_{1214} = R_{1434} = R_{1234} \\ =& R_{2334} = R_{2314}, \\ R_{1213} =& R_{1224} = R_{1413} = R_{2414} = R_{2423} = R_{2313} = R_{1334} = R_{2434}. \end{aligned}$$

By applying the Bianchi identity to the above components of R, we obtain (3.1).

Vice versa, from (2.1) and (3.1) it follows (3.2), so (2.15) holds true. □

Proposition 3.2. *If a manifold (M, g, S) has the property (2.15), then the components of the Ricci tensor ρ satisfy*

$$\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44}, \quad \rho_{12} = \rho_{23} = \rho_{34} = -\rho_{14}, \quad \rho_{13} = \rho_{24} = 0. \tag{3.3}$$

Proof. Due to Proposition 3.1, the components of the curvature tensor R satisfy (3.1). For brevity, we denote

$$R_1 = R_{1313}, \quad R_2 = R_{1213}. \tag{3.4}$$

Thus, having in mind (2.8), (2.12), (3.1) and (3.4), we get the components of ρ as follows:

$$\begin{aligned} \rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} &= \frac{2}{A^2 - 2B^2}\big(2BR_2 - AR_1\big), \\ \rho_{12} = \rho_{23} = \rho_{34} = -\rho_{14} &= \frac{2}{A^2 - 2B^2}\big(BR_1 - AR_2\big), \\ \rho_{13} = \rho_{24} = 0. & \end{aligned} \tag{3.5}$$

So the equalities (3.3) are valid. □

A Riemannian manifold is said to be Einstein if its Ricci tensor ρ is a constant multiple of the metric tensor g, i.e.

$$\rho(x, y) = \alpha g(x, y). \tag{3.6}$$

In [17], for locally decomposable Riemannian manifolds is defined a class of almost Einstein manifolds. For the considered in our paper manifolds, we give the following

Definition 3.3. A Riemannian manifold (M, g, S) is called almost Einstein if the metrics g and $\tilde{g}$ satisfy

$$\rho(x, y) = \alpha g(x, y) + \beta \tilde{g}(x, y), \tag{3.7}$$

where α and β are smooth functions on M.

Theorem 3.4. *The manifold (M, g, S) with the property (2.15) is almost Einstein.*

Proof. According to Proposition 3.2, for (M, g, S) the equalities (3.3) are valid. Consequently, using (2.8), (2.9), (2.13) and (3.3), we get the values of the scalar curvature τ and τ^* as follows:

$$\tau = \frac{4}{A^2 - 2B^2}\big(A\rho_{11} - 2B\rho_{12}\big), \quad \tau^* = \frac{4}{A^2 - 2B^2}\big(A\rho_{12} - B\rho_{11}\big).$$

Immediately from the latter equalities we have

$$\rho_{11} = \frac{\tau}{4}A + \frac{2\tau^*}{4}B, \quad \rho_{12} = \frac{\tau}{4}B + \frac{\tau^*}{4}A, \tag{3.8}$$

and bearing in mind (2.4) and (2.7) we get

$$\rho_{11} = \frac{\tau}{4}g_{11} + \frac{\tau^*}{4}\tilde{g}_{11}, \quad \rho_{12} = \frac{\tau}{4}g_{12} + \frac{\tau^*}{4}\tilde{g}_{12}.$$

Then, taking into account (2.4), (2.7), (3.3) and (3.8), we obtain

$$\rho_{ij} = \frac{\tau}{4}g_{ij} + \frac{\tau^*}{4}\tilde{g}_{ij}, \tag{3.9}$$

i.e.

$$\rho(x, y) = \frac{\tau}{4}g(x, y) + \frac{\tau^*}{4}\tilde{g}(x, y). \tag{3.10}$$

Therefore, comparing (3.10) with (3.7), we state that (M, g, S) is an almost Einstein manifold. □

Let (M, g, S) satisfy the conditions of Theorem 3.4. If we suppose that (M, g, S) is an Einstein manifold, then its Ricci tensor ρ has the form (3.6). Hence (3.10) implies the following

Corollary 3.5. *If the manifold (M, g, S) with the property (2.15) is Einstein then*

$$\tau^* = 0. \tag{3.11}$$

In the next theorem, we express the curvature tensor R of an almost Einstein manifold (M, g, S) by both structures g and S.

Theorem 3.6. *Let (M, g, S) have the property (2.15). Then the curvature tensor R has the form*

$$R = \frac{\tau}{16}\big(2\pi_1 + \pi_3\big) + \frac{\tau^*}{8}\pi_2, \tag{3.12}$$

where

$$\begin{aligned} \pi_1(x, y, z, u) &= g(y, z)g(x, u) - g(x, z)g(y, u), \\ \pi_2(x, y, z, u) &= g(y, z)\tilde{g}(x, u) + g(x, u)\tilde{g}(y, z) \\ &\quad - g(x, z)\tilde{g}(y, u) - g(y, u)\tilde{g}(x, z), \\ \pi_3(x, y, z, u) &= \tilde{g}(y, z)\tilde{g}(x, u) - \tilde{g}(x, z)\tilde{g}(y, u). \end{aligned} \tag{3.13}$$

Proof. Due to Proposition 3.2, the components of the Ricci tensor ρ of (M, g, S) are given by (3.5). Therefore, by straightforward computation, we get

$$R_1 = -\frac{1}{2}(A\rho_{11} + 2B\rho_{12}) \qquad R_2 = -\frac{1}{2}(B\rho_{11} + A\rho_{12}).$$

We substitute (3.8) into the above equalities and obtain

$$\begin{aligned} R_1 &= -\frac{1}{8}\big((A^2 + 2B^2)\tau + 4AB\tau^*\big), \\ R_2 &= -\frac{1}{8}\big(2AB\tau + (2B^2 + A^2)\tau^*\big). \end{aligned} \tag{3.14}$$

From (2.4), (2.7), (3.4) and (3.14) it follows

$$\begin{aligned} R_{1313} &= \frac{\tau}{16}\Big(2(g_{13}g_{31} - g_{11}g_{33}) + \tilde{g}_{13}\tilde{g}_{31} - \tilde{g}_{11}\tilde{g}_{33}\Big) \\ &\quad + \frac{\tau^*}{8}\big(g_{13}\tilde{g}_{31} + \tilde{g}_{13}g_{31} - \tilde{g}_{11}g_{33} - g_{11}\tilde{g}_{33}\big), \\ R_{1213} &= \frac{\tau}{16}\Big(2(g_{13}g_{21} - g_{11}g_{23}) + \tilde{g}_{13}\tilde{g}_{21} - \tilde{g}_{11}\tilde{g}_{23}\Big) \\ &\quad + \frac{\tau^*}{8}\big(g_{13}\tilde{g}_{21} + \tilde{g}_{13}g_{21} - \tilde{g}_{11}g_{23} - g_{11}\tilde{g}_{23}\big). \end{aligned}$$

Consequently, using (2.4), (2.7), (3.1), (3.4) and (3.14), we have

$$\begin{aligned} R_{ijkh} &= \frac{\tau}{16}\Big(2(g_{ih}g_{jk} - g_{ik}g_{jh}) + \tilde{g}_{ih}\tilde{g}_{jk} - \tilde{g}_{ik}\tilde{g}_{jh}\Big) \\ &\quad + \frac{\tau^*}{8}\big(g_{ih}\tilde{g}_{jk} + \tilde{g}_{ih}g_{jk} - \tilde{g}_{ik}g_{jh} - g_{ik}\tilde{g}_{jh}\big), \end{aligned}$$

which is a local form of (3.12) with (3.13). □

4. Orthogonal S-basis of T_pM

If x is a vector in a tangent space T_pM of (M, g, S), then applying (2.1) we get the system of vectors $\{S^3x, S^2x, Sx, x\}$. We will use a basis and an orthogonal basis of the type $\{S^3x, S^2x, Sx, x\}$ in T_pM. Therefore, in this section we will consider the existence of such bases.

If x is a nonzero vector on (M, g, S), then according to (2.1) we have $Sx \neq \pm x$. Thus the angle φ between x and Sx belongs to the interval $(0, \pi)$. Evidently, the vectors x, Sx, S^2x and S^3x determine six angles, which belong to $(0, \pi)$. For these angles we establish the next statement.

Theorem 4.1. *Let x be a nonzero vector on (M, g, S). Then*

$$\begin{aligned} &\angle(x, Sx) = \angle(Sx, S^2x) = \angle(S^2x, S^3x) = \varphi, \quad \angle(x, S^3x) = \pi - \varphi, \\ &\angle(x, S^2x) = \angle(Sx, S^3x) = \frac{\pi}{2}, \end{aligned} \tag{4.1}$$

where $\varphi \in (0, \pi)$.

Proof. Let $x = (x^1, x^2, x^3, x^4)$ be a nonzero vector on (M, g, S). By using (2.1), we get

$$\begin{aligned} Sx &= (x^2, x^3, x^4, -x^1),\ \ S^2x = (x^3, x^4, -x^1, -x^2),\\ S^3x &= (x^4, -x^1, -x^2, -x^3). \end{aligned} \tag{4.2}$$

Having in mind the components of x, also (2.4) and (4.2), we calculate

$$\begin{aligned} g(x,x) =& A\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big)\\ &+ 2B(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4),\\ g(x,Sx) =& A(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4)\\ &+ B\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big). \end{aligned} \tag{4.3}$$

From (2.2) and (2.3) it follows

$$g(x, Sx) = -g(x, S^3x), \qquad g(x, S^2x) = 0. \tag{4.4}$$

Now, due to (2.3) and (2.5), we can determine the angle φ between x and Sx, and the angle ϕ between x and S^2x as follows:

$$\cos\varphi = \frac{g(x, Sx)}{g(x,x)}, \qquad \cos\phi = \frac{g(x, S^2x)}{g(x,x)}. \tag{4.5}$$

We apply (4.3) and (4.4) in (4.5) and find

$$\cos\varphi = \frac{A(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4) + B\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big)}{A\big((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\big) + 2B(x^1x^2 + x^2x^3 + x^3x^4 - x^1x^4)}$$

$$\cos\phi = 0.$$

Then, bearing in mind (2.3) and (4.4), we get (4.1). □

Definition 4.2. A basis of type $\{S^3x, S^2x, Sx, x\}$ of T_pM is called an *S-basis*. In this case we say that *the vector x induces an S-basis of T_pM*.

The following statements hold.

Theorem 4.3. *Every nonzero vector $x = (x^1, x^2, x^3, x^4)$, which satisfies*

$$\begin{aligned} 4x^2x^4\big((x^1)^2 - (x^3)^2\big) &+ 4x^1x^3\big((x^4)^2 - (x^2)^2\big)\\ &+ \big((x^1)^2 + (x^3)^2\big)^2 + \big((x^2)^2 + (x^4)^2\big)^2 \neq 0, \end{aligned} \tag{4.6}$$

induces an S-basis of T_pM.

Proof. If a nonzero vector $x \in T_pM$ has coordinates (x^1, x^2, x^3, x^4), then using (4.2) we get the determinant formed by the coordinates of the vectors x, Sx, S^2x and S^3x. It is

$$\begin{aligned} \triangle = 4x^2x^4\big((x^1)^2 - (x^3)^2\big) &+ 4x^1x^3\big((x^4)^2 - (x^2)^2\big)\\ &+ \big((x^1)^2 + (x^3)^2\big)^2 + \big((x^2)^2 + (x^4)^2\big)^2. \end{aligned}$$

In case that (4.6) is valid, we have $\triangle \neq 0$, which implies that x, Sx, S^2x and S^3x form a basis. □

Lemma 4.4. *Let a vector x induce an S-basis and let φ be the angle between x and Sx. The following inequalities are valid:*

$$\frac{\pi}{4} < \varphi < \frac{3\pi}{4}. \tag{4.7}$$

Proof. We suppose without loss of generality that $g(x,x) = 1$. Thus, because of (2.3), (4.4) and (4.5), we find

$$\begin{aligned} g(x,Sx) = g(Sx,S^2x) = g(S^2x,S^3x) = -g(x,S^3x) = \cos\varphi, \\ g(x,S^2x) = g(Sx,S^3x) = 0. \end{aligned} \tag{4.8}$$

We consider a nonzero vector y, such that

$$y = -\cos\varphi x + Sx - \cos\varphi S^2x. \tag{4.9}$$

Since g is a Riemannian metric we have $g(y,y) > 0$. Substituting (4.9) into the latter inequality, and using (4.8), we get

$$1 - 2\cos^2\varphi > 0.$$

Then, taking into account $0 < \varphi < \pi$, we obtain (4.7). □

According to Theorem 4.3, there are many S-bases of T_pM. Hence, bearing in mind Theorem 4.1 and Lemma 4.4, we arrive at the following

Theorem 4.5. *For every manifold (M,g,S) there exists an orthogonal S-basis of T_pM.*

5. Curvature properties of (M,g,S)

The sectional curvature of a non-degenerate 2-plane $\{x,y\}$ spanned by the vectors $x,y \in T_pM$ is the value

$$k(x,y) = \frac{R(x,y,x,y)}{g(x,x)g(y,y) - g^2(x,y)}. \tag{5.1}$$

Let a vector x induce an S-basis of T_pM for (M,g,S). There are determined six 2-planes $\{x,Sx\}$, $\{x,S^2x\}$, $\{x,S^3x\}$, $\{Sx,S^2x\}$, $\{Sx,S^3x\}$ and $\{S^2x,S^3x\}$ in T_pM. For the angles between the pairs of vectors equalities (4.1) are valid. Moreover, the angle $\varphi = \angle(x,Sx)$ satisfies (4.7). In the next theorem we establish the relations among the sectional curvatures of the 2-planes generated by an S-basis, the angle φ, the scalar curvature τ and τ^*.

Theorem 5.1. *Let (M,g,S) have the property (2.15) and let a vector x induce an S-basis. Then the sectional curvatures of the 2-planes, determined by the*

S-basis, are

$$\begin{aligned} k(x,Sx) &= k(Sx,S^2x) = k(x,S^3x) = k(S^2x,S^3x)\\ &= \frac{1}{16(\cos^2\varphi-1)}\Big(\tau(1+2\cos^2\varphi)+4\tau^*\cos\varphi\Big),\\ k(x,S^2x) &= k(Sx,S^3x) = -\frac{1}{8}\Big(\tau(1+2\cos^2\varphi)+4\tau^*\cos\varphi\Big), \end{aligned} \tag{5.2}$$

where $\varphi = \angle(x, Sx)$.

Proof. Let a vector x induce an S-basis. The equalities (2.3), (4.4) and (4.5) imply

$$\begin{aligned} &g(x,Sx) = g(Sx,S^2x) = g(S^2x,S^3x) = -g(x,S^3x) = g(x,x)\cos\varphi,\\ &g(x,S^2x) = g(Sx,S^3x) = 0. \end{aligned} \tag{5.3}$$

Hence, from (2.2), (2.3), (2.6) and (5.3), we find

$$\begin{aligned} &\tilde{g}(x,x) = 2g(x,x)\cos\varphi, \quad \tilde{g}(x,S^2x) = 0,\\ &\tilde{g}(x,Sx) = -\tilde{g}(x,S^3x) = g(x,x). \end{aligned} \tag{5.4}$$

Applying (3.12), (3.13), (5.3) and (5.4) in (5.1), we obtain (5.2). □

Corollary 5.2. *Let a vector x induce an orthonormal S-basis. Then*

$$\begin{aligned} k(x,Sx) =& k(Sx,S^2x) = k(x,S^3x) = k(S^2x,S^3x) = -\frac{\tau}{16},\\ k(x,S^2x) =& k(Sx,S^3x) = -\frac{\tau}{8}. \end{aligned}$$

Proof. The proof follows directly from (5.2), when $\varphi = \frac{\pi}{2}$. □

Due to Theorem 5.1 and Corollary 3.5 we establish the following

Proposition 5.3. *If (M, g, S) with (2.15) is an Einstein manifold, then the sectional curvatures of the 2-planes, determined by an S-basis, are*

$$\begin{aligned} k(x,Sx) =& k(Sx,S^2x) = k(x,S^3x) = k(S^2x,S^3x) = \frac{\tau(1+2\cos^2\varphi)}{16(\cos^2\varphi-1)},\\ k(x,S^2x) =& k(Sx,S^3x) = -\frac{\tau}{8}(1+2\cos^2\varphi). \end{aligned}$$

Now, we recall that the Ricci curvature in the direction of a nonzero vector x is the value

$$r(x) = \frac{\rho(x,x)}{g(x,x)}. \tag{5.5}$$

Theorem 5.4. *Let (M, g, S) have the property (2.15). If a vector x induces an S-basis, then the Ricci curvatures in the direction of the basis vectors are*

$$r(x) = r(Sx) = r(S^2x) = r(S^3x) = \frac{\tau}{4} + \frac{\tau^*}{2}\cos\varphi, \tag{5.6}$$

where $\varphi = \angle(x, Sx)$.

Proof. In the course of the proof of Theorem 3.4, we find that ρ is given by (3.10). Then, using (2.3), we obtain

$$\begin{aligned}\rho(x,x)=\rho(Sx,Sx)=\rho(S^2x,S^2x)=\rho(S^3x,S^3x)\\ =\frac{\tau}{4}g(x,x)+\frac{\tau^*}{4}\tilde{g}(x,x).\end{aligned} \tag{5.7}$$

Let a vector x induce an S-basis. From (2.3), (5.4), (5.5) and (5.7) it follows (5.6). □

Proposition 5.5. *Let (M,g,S) with (2.15) be an Einstein manifold. If a vector x induces an S-basis, then the Ricci curvatures in the direction of the basis vectors are*

$$r(x)=r(Sx)=r(S^2x)=r(S^3x)=\frac{\tau}{4}.$$

Proof. The above equalities follow directly by substituting $\tau^*=0$ into (5.6). □

6. Manifolds with parallel structures

In this section we study a manifold (M,g,S), whose structure S satisfies (2.14). Also, we consider an associated manifold (M,g,J) with a structure $J=S^2$. Bearing in mind (2.1) and (2.3), we get that the manifold (M,g,J) is Hermitian and the structure J is complex. In case that J is parallel (M,g,J) is a Kähler manifold. The characteristic condition of a Kähler manifold is

$$\nabla J=0. \tag{6.1}$$

Evidently, for the structure $J=S^2$, the equality (2.14) implies (6.1).

Theorem 6.1. *Let (M,g,S) have the property (2.14). Then the scalar curvature τ and τ^* satisfy*

$$3\tau_1=\tau^*_2-\tau^*_4,\quad 3\tau_2=\tau^*_1+\tau^*_3,\quad 3\tau_3=\tau^*_2+\tau^*_4,\quad 3\tau_4=-\tau^*_1+\tau^*_3, \tag{6.2}$$

where $\tau_i=\frac{\partial\tau}{\partial X^i}$, $\tau^*_i=\frac{\partial\tau^*}{\partial X^i}$.

Proof. It is known that in a Riemannian manifold for the scalar curvature τ and the Ricci tensor ρ it is valid

$$\nabla_i\rho^i_k=\frac{1}{2}\nabla_k\tau, \tag{6.3}$$

where $\rho^i_k=\rho_{ak}g^{ai}$.

On the other hand, if (M,g,S) satisfies (2.14), then it satisfies (2.15). Therefore, the Ricci tensor has the expression (3.9). Hence, from (2.1), (2.4), (2.7), (2.8) and (3.9), we get

$$\rho^i_k=\frac{\tau}{4}\delta^i_k+\frac{\tau^*}{4}\big(S^i_k-(S^i_k)^3\big),$$

where δ^i_k are the Kronecker symbols. Using the above equalities, (2.14) and (6.3) we obtain

$$\tau_k = \frac{\tau_i}{4}\delta^i_k + \frac{\tau^*_i}{4}\big(S^i_k - (S^i_k)^3\big),$$

where because of (2.1) it follows (6.2). □

6.1. Conditions for parallel structures

Theorem 6.2. *The manifold* (M, g, S) *satisfies* (2.14) *if and only if*

$$A_1 = B_2 - B_4, \quad A_2 = B_1 + B_3, \quad A_3 = B_2 + B_4, \quad A_4 = B_3 - B_1, \tag{6.4}$$

where $A_i = \dfrac{\partial A}{\partial X^i}$, $B_i = \dfrac{\partial B}{\partial X^i}$.

Proof. If Γ^s_{ij} are the Christoffel symbols of ∇, then

$$\nabla_i S^t_j = \partial_i S^t_j + \Gamma^t_{ik}S^k_j - \Gamma^k_{ij}S^t_k. \tag{6.5}$$

Together with (2.14), (6.5) yields

$$\Gamma^t_{ik}S^k_j = \Gamma^k_{ij}S^t_k. \tag{6.6}$$

From (2.1) and (6.6) we get

$$\begin{aligned}
&\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \Gamma^3_{22} = \Gamma^4_{23} = -\Gamma^1_{24} = -\Gamma^1_{33} = -\Gamma^2_{34} = -\Gamma^3_{44},\\
&\Gamma^2_{11} = \Gamma^3_{12} = \Gamma^4_{13} = -\Gamma^1_{14} = \Gamma^4_{22} = -\Gamma^1_{23} = -\Gamma^2_{24} = -\Gamma^2_{33} = -\Gamma^3_{34} = -\Gamma^4_{44},\\
&\Gamma^3_{11} = \Gamma^4_{12} = -\Gamma^1_{13} = -\Gamma^2_{14} = -\Gamma^1_{22} = -\Gamma^2_{23} = -\Gamma^3_{24} = -\Gamma^3_{33} = -\Gamma^4_{34} = \Gamma^1_{44},\\
&\Gamma^4_{11} = -\Gamma^1_{12} = -\Gamma^2_{13} = -\Gamma^3_{14} = -\Gamma^2_{22} = -\Gamma^3_{23} = -\Gamma^4_{24} = -\Gamma^4_{33} = \Gamma^1_{34} = \Gamma^2_{44}.
\end{aligned}$$

Then, applying (2.4) and (2.8) in the well-known identities

$$2\Gamma^s_{ij} = g^{as}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}), \tag{6.7}$$

we obtain conditions (6.4).

Vice versa. From (2.1), (2.4), (2.8), (6.4) and (6.7) it follows (6.6). Consequently, by (2.1), (6.5) and (6.6) we get (2.14). □

Theorem 6.3. *The manifold* (M, g, J) *is Kähler if and only if the equalities* (6.4) *are valid.*

Proof. Having in mind (2.1), we get that the components of the structure $J = S^2$ on (M, g, J) are given by the skew-circulant matrix

$$(J^k_j) = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{6.8}$$

Let (M, g, J) be a Kähler manifold. Therefore, from (6.1), (6.8) and

$$\nabla_i J^t_j = \partial_i J^t_j + \Gamma^t_{ik}J^k_j - \Gamma^k_{ij}J^t_k$$

it follows

$$\Gamma_{ik}^{t} J_{j}^{k} = \Gamma_{ij}^{k} J_{k}^{t}. \tag{6.9}$$

Together with (6.8), (6.9) yields

$$\begin{aligned}
&\Gamma_{11}^{1} = \Gamma_{13}^{3} = -\Gamma_{33}^{1}, \Gamma_{14}^{4} = \Gamma_{23}^{4} = \Gamma_{12}^{2} = -\Gamma_{34}^{2}, \Gamma_{22}^{3} = -\Gamma_{24}^{1} = -\Gamma_{44}^{3},\\
&\Gamma_{11}^{2} = \Gamma_{13}^{4} = -\Gamma_{33}^{2}, \Gamma_{14}^{1} = \Gamma_{23}^{1} = -\Gamma_{12}^{3} = \Gamma_{34}^{3}, \Gamma_{22}^{4} = -\Gamma_{24}^{2} = -\Gamma_{44}^{4},\\
&\Gamma_{11}^{3} = -\Gamma_{13}^{1} = -\Gamma_{33}^{3}, \Gamma_{14}^{2} = \Gamma_{23}^{2} = -\Gamma_{12}^{4} = \Gamma_{34}^{4}, \Gamma_{22}^{1} = \Gamma_{24}^{3} = -\Gamma_{44}^{1},\\
&\Gamma_{11}^{4} = -\Gamma_{13}^{2} = -\Gamma_{33}^{4}, \Gamma_{14}^{3} = \Gamma_{23}^{3} = \Gamma_{12}^{1} = -\Gamma_{34}^{1}, \Gamma_{22}^{2} = \Gamma_{24}^{4} = -\Gamma_{44}^{2}.
\end{aligned}$$

From the above equalities, using (2.4), (2.8) and (6.7), we get conditions (6.4).

Vice versa. From (6.4) it follows (2.14) and hence (6.1). So J is a parallel structure. □

Bearing in mind Theorems 6.2 and 6.3 we state the following

Corollary 6.4. *The structure S of (M, g, S) is parallel with respect to ∇ if and only if the structure J of (M, g, J) is parallel with respect to ∇.*

7. Lie groups as 4-dimensional Riemannian manifolds with skew-circulant structures

Let G be a 4-dimensional real connected Lie group and $\mathfrak{g}$ be its Lie algebra with a basis $\{x_1, x_2, x_3, x_4\}$. We introduce a tensor structure S and a left invariant metric g as follows:

$$Sx_1 = -x_4,\ Sx_2 = x_1,\ Sx_3 = x_2,\ Sx_4 = x_3, \tag{7.1}$$

$$g(x_i, x_j) = \begin{cases} 0, & i \neq j;\\ 1, & i = j. \end{cases} \tag{7.2}$$

Obviously (2.2) and (2.3) are valid. Therefore (G, g, S) is a Riemannian manifold of the considered type.

If we suppose that S is an Abelian structure on a Lie group G, then the commutators $[x_i, x_j]$ satisfy

$$[x_i, x_j] = [Sx_i, Sx_j]. \tag{7.3}$$

The conditions (7.1), (7.3) and the Jacobi identity for $[x_i, x_j]$ imply

$$\begin{aligned}
&[x_1, x_2] = [x_1, x_4] = [x_2, x_3] = [x_3, x_4] = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4,\\
&[x_1, x_3] = [x_2, x_4] = (\lambda_2 - \lambda_4)x_1 + (\lambda_1 + \lambda_3)x_2 + (\lambda_2 + \lambda_4)x_3\\
&\qquad + (\lambda_3 - \lambda_1)x_4,
\end{aligned} \tag{7.4}$$

where $\lambda_i \in \mathbb{R}$.

In this section we investigate a manifold (G, g, S) with a Lie algebra $\mathfrak{g}$ determined by (7.4), i.e., a manifold (G, g, S) with an Abelian structure S.

Theorem 7.1. *Let (G, g, S) be a manifold with a Lie algebra $\mathfrak{g}$ determined by (7.4). Then (G, g, S) has the property (2.14).*

Proof. The well-known Koszul formula implies

$$2g(\nabla_{x_i}x_j, x_k) = g([x_i, x_j], x_k) + g([x_k, x_i], x_j) + g([x_k, x_j], x_i),$$

and having in mind (7.2) and (7.4), we find

$$\begin{aligned}
\nabla_{x_1}x_1 &= -\lambda_1(x_2 + x_4) + (\lambda_4 - \lambda_2)x_3,\\
\nabla_{x_1}x_2 &= \lambda_1(x_1 - x_3) + (\lambda_4 - \lambda_2)x_4,\\
\nabla_{x_1}x_3 &= \lambda_1(x_2 - x_4) + (\lambda_2 - \lambda_4)x_1,\\
\nabla_{x_1}x_4 &= \lambda_1(x_1 + x_3) + (\lambda_2 - \lambda_4)x_2,\\
\nabla_{x_2}x_1 &= -\lambda_2(x_2 + x_4) - (\lambda_1 + \lambda_3)x_3,\\
\nabla_{x_2}x_2 &= \lambda_2(x_1 - x_3) - (\lambda_1 + \lambda_3)x_4,\\
\nabla_{x_2}x_3 &= \lambda_2(x_2 - x_4) + (\lambda_1 + \lambda_3)x_1,\\
\nabla_{x_2}x_4 &= \lambda_2(x_1 + x_3) + (\lambda_1 + \lambda_3)x_2,\\
\nabla_{x_3}x_1 &= -\lambda_3(x_2 + x_4) - (\lambda_2 + \lambda_4)x_3,\\
\nabla_{x_3}x_2 &= \lambda_3(x_1 - x_3) - (\lambda_2 + \lambda_4)x_4,\\
\nabla_{x_3}x_3 &= \lambda_3(x_2 - x_4) + (\lambda_2 + \lambda_4)x_1,\\
\nabla_{x_3}x_4 &= \lambda_3(x_1 + x_3) + (\lambda_2 + \lambda_4)x_2,\\
\nabla_{x_4}x_1 &= -\lambda_4(x_2 + x_4) + (\lambda_1 - \lambda_3)x_3,\\
\nabla_{x_4}x_2 &= \lambda_4(x_1 - x_3) + (\lambda_1 - \lambda_3)x_4,\\
\nabla_{x_4}x_3 &= \lambda_4(x_2 - x_4) + (\lambda_3 - \lambda_1)x_1,\\
\nabla_{x_4}x_4 &= \lambda_4(x_1 + x_3) + (\lambda_3 - \lambda_1)x_2. \qquad (7.5)
\end{aligned}$$

From (7.1), (7.5) and the formula $(\nabla_{x_i}S)x_j = \nabla_{x_i}Sx_j - S\nabla_{x_i}x_j$ we get $(\nabla_{x_i}S)x_j = 0$, i.e. (2.14) is valid. □

Further, using (2.10), (2.11), (7.2), (7.4) and (7.5) we calculate the following components of the curvature tensor R:

$$\begin{aligned}
R_{1313} &= R_{2424} = R_{1324} = 2R_{1212} = 2R_{1414} = 2R_{2323} = 2R_{3434}\\
&= 2R_{1223} = 2R_{1214} = 2R_{1434} = 2R_{1234} = 2R_{2334} = 2R_{2314}\\
&= 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2),\\
R_{1213} &= R_{1224} = R_{1413} = R_{2414} = R_{2423} = R_{2313} = R_{1334} = R_{2434}\\
&= 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 - \lambda_1\lambda_4). \qquad (7.6)
\end{aligned}$$

The rest of the nonzero components are obtained from the properties

$$R_{ijks} = R_{ksij}, \ R_{ijks} = -R_{jiks} = -R_{ijsk}.$$

From (7.2), (7.6) and the formula (2.12) we get the components of the Ricci tensor ρ:

$$\begin{aligned}&\rho_{11}=\rho_{22}=\rho_{33}=\rho_{44}=-4(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2),\\&\rho_{12}=\rho_{23}=\rho_{34}=-4(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_4-\lambda_1\lambda_4),\\&\rho_{13}=\rho_{24}=0,\quad \rho_{14}=-\rho_{12}.\end{aligned}\tag{7.7}$$

Now, using (7.1) and (7.2) we find the components of $\tilde{g}$ determined by (2.6), and the components of its inverse. They are as follows:

$$\begin{aligned}&\tilde{g}_{11}=\tilde{g}_{22}=\tilde{g}_{33}=\tilde{g}_{44}=0,\tilde{g}_{12}=\tilde{g}_{23}=\tilde{g}_{34}=-\tilde{g}_{14}=1,\tilde{g}_{13}=\tilde{g}_{24}=0,\\&\tilde{g}^{11}=\tilde{g}^{22}=\tilde{g}^{33}=\tilde{g}^{44}=0,\tilde{g}^{12}=\tilde{g}^{23}=\tilde{g}^{34}=-\tilde{g}^{14}=\frac{1}{2},\tilde{g}^{13}=\tilde{g}^{24}=0.\end{aligned}$$

Then, from (2.13), (7.2) and (7.7), we get the values of the scalar curvature τ and τ^* as follows:

$$\tau=-16(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2),\ \ \tau^*=-16(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_4-\lambda_1\lambda_4).\tag{7.8}$$

Consequently, the components of g and ρ, the values of τ and τ^*, given by (7.2), (7.7) and (7.8) respectively, satisfy (3.9), i.e., (G,g,S) is an almost Einstein manifold.

Further, from (5.1), (7.2) and (7.6), for the sectional curvatures of the basic 2-planes we find

$$\begin{aligned}&k(x_2,x_4)=k(x_1,x_3)=2(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2),\\&k(x_1,x_2)=k(x_1,x_4)=k(x_2,x_3)=k(x_3,x_4)=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2.\end{aligned}\tag{7.9}$$

Therefore, we arrive at the following

Theorem 7.2. *Let (G,g,S) be a manifold with a Lie algebra $\mathfrak{g}$ determined by (7.4). Then*

(i) *the nonzero components of the curvature tensor R are (7.6);*
(ii) *the components of the Ricci tensor ρ are (7.7);*
(iii) *the scalar curvature τ and τ^* are (7.8). The manifold is almost Einstein;*
(iv) *the sectional curvatures of the basic 2-planes are (7.9).*

7.1. Einstein manifolds

Let G' be a subgroup of G, where (G,g,S) is a manifold with a Lie algebra $\mathfrak{g}$ determined by (7.4). Let (G',g,S) be an Einstein manifold. Bearing in mind Corollary 3.5 and the second equality of (7.8) we construct two examples of such a manifold.

Case (A) $\lambda_3=\lambda_1,\quad \lambda_2=0,$

Case (B) $\lambda_1=\lambda_2+\lambda_4,\quad \lambda_3=\lambda_4-\lambda_2.$

We note that these cases exhaust the set of Einstein manifolds (G',g,S) with an Abelian structure S.

Let us consider the case (A). With the help of (7.4), (7.7), (7.8) and (7.9), we prove the following

Proposition 7.3. *Let (G', g, S) be a manifold with a Lie algebra $\mathfrak{g}$ determined by*

$$[x_1, x_2] = [x_1, x_4] = [x_2, x_3] = [x_3, x_4] = \lambda_1 x_1 + \lambda_1 x_3 + \lambda_4 x_4,$$
$$[x_1, x_3] = [x_2, x_4] = -\lambda_4 x_1 + 2\lambda_1 x_2 + \lambda_4 x_3.$$

Then

(i) *the nonzero components of ρ are $\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} = -4(2\lambda_1^2 + \lambda_4^2)$;*
(ii) *the scalar curvature is $\tau = -16(2\lambda_1^2 + \lambda_4^2)$;*
(iii) *the sectional curvatures of the basic 2-planes are*

$$k(x_2, x_4) = k(x_1, x_3) = 2(2\lambda_1^2 + \lambda_4^2),$$
$$k(x_1, x_2) = k(x_1, x_4) = k(x_2, x_3) = k(x_3, x_4) = 2\lambda_1^2 + \lambda_4^2.$$

For the case (B), with similar calculations, we establish the following

Proposition 7.4. *Let (G', g, S) be a manifold with a Lie algebra $\mathfrak{g}$ determined by*

$$[x_1, x_2] = [x_1, x_4] = [x_2, x_3] = [x_3, x_4] = (\lambda_2 + \lambda_4)x_1 + \lambda_2 x_2 + (\lambda_4 - \lambda_2)x_3 + \lambda_4 x_4,$$
$$[x_1, x_3] = [x_2, x_4] = (\lambda_2 - \lambda_4)x_1 + 2\lambda_4 x_2 + (\lambda_2 + \lambda_4)x_3 - 2\lambda_2 x_4.$$

Then

(i) *the nonzero components of ρ are $\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} = -12(\lambda_2^2 + \lambda_4^2)$;*
(ii) *the scalar curvature is $\tau = -48(\lambda_2^2 + \lambda_4^2)$;*
(iii) *the sectional curvatures of the basic 2-planes are*

$$k(x_2, x_4) = k(x_1, x_3) = 6(\lambda_2^2 + \lambda_4^2),$$
$$k(x_1, x_2) = k(x_1, x_4) = k(x_2, x_3) = k(x_3, x_4) = 3(\lambda_2^2 + \lambda_4^2).$$

Conclusion

In fact, we investigate two classes of manifolds (M, g, S). The wider class consists manifolds with the property (2.15). The manifolds with a parallel structure S belong to the narrower class. In both classes Einstein and almost Einstein manifolds are determined. In both classes curvature properties of (M, g, S) are obtained. Examples of manifolds with a parallel structure S are constructed on Lie groups. Our future problem is to construct an example of a manifold (M, g, S) which satisfies (2.15), but does not satisfy (2.14).

Acknowledgements

The authors are grateful to Prof. G. Dzhelepov for his valuable comments.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Iva Dokuzova
Department of Algebra and Geometry
University of Plovdiv Paisii Hilendarski
24 Tzar Asen
4000 Plovdiv
Bulgaria
e-mail: dokuzova@uni-plovdiv.bg

Dimitar Razpopov
Department of Mathematics and Physics
Agricultural University of Plovdiv
12 Mendeleev Blvd.
4000 Plovdiv
Bulgaria
e-mail: razpopov@au-plovdiv.bg

Received: August 27, 2019.
Revised: December 6, 2019.