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The value of the work done by an isotropic vector force field along an isotropic curve

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Abstract. In the present paper we consider a 3-dimensional differentiable manifold M equipped with a Riemannian metric g and an endomorphism Q, whose third power is the identity and Q acts as an isometry on g. Both structures g and Q determine an associated metric f on (M, g, Q). The metric f is necessary indefinite and it defines isotropic vectors in the tangent space T_pM at an arbitrary point p on M.

The physical forces are represented by vector fields. We investigate physical forces whose vectors are in T_pM on (M, g, Q). Moreover, these vectors are isotropic and they act along isotropic curves. We study the physical work done by such forces.

1. Introduction

The physical work and the physical force on differentiable manifolds have a great application in physics. Vector fields are often used to model a force, such as the magnetic or gravitational force, as it changes from one point to another point. As a particle moves through a force field along a curve c, the work done by the force is the product of force and displacement. There are some papers concerning physical results on light-like (degenerate) objects of differentiable manifolds ([4], [8] and [9]).

The object of the present paper is a 3-dimensional differentiable manifold M equipped with a Riemannian metric g and a tensor Q of type $(1, 1)$, whose third power is the identity and Q acts as an isometry on g. Such a manifold (M, g, Q) is defined in [6] and studied in [1], [2], [3] and [7]. Also, we consider an associated metric f, which is introduced in [7]. The metric f is necessary indefinite and it determines space-like vectors, isotropic vectors and time-like vectors in the tangent space T_pM at an arbitrary point p on M.

We investigate physical forces whose vectors are in T_pM on (M, g, Q). Moreover, these vectors are isotropic with respect to f and they act along isotropic curves. We study the physical work done by such forces.

2. Preliminaries

Let M be a 3-dimensional Riemannian manifold equipped with an endomorphism Q in the tangent space T_pM, $p \in M$. Let the local coordinates of Q with respect to some coordinate system form the circulant matrix:

$$(Q_i^j) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then Q has the property

$$Q^3 = \mathrm{id}. \tag{1}$$

Let g be a positive definite metric on M, which satisfies the equality

$$g(Qr, Qi) = g(r, i). \tag{2}$$

In (2) and further r, i, w will stand for arbitrary vectors in T_pM.

Such a manifold (M, g, Q) is introduced in [6].

It is well-known that the norm of every vector i is given by $\|i\| = \sqrt{g(i,i)}$. Then, having in mind (2), for the angle $\varphi = \angle(i, Qi)$ we have

$$\cos\varphi = \frac{g(i, Qi)}{g(i, i)}.$$

In [6], for (M, g, Q), it is verified that the angle φ is in $[0, \frac{2\pi}{3}]$. If $\varphi \in (0, \frac{2\pi}{3})$, then the vector i form a basis $\{i, Qi, Q^2i\}$, which is called a Q-basis of T_pM.

The associated metric f on (M, g, Q), determined by

$$f(r, i) = g(r, Qi) + g(Qr, i). \tag{3}$$

is necessary indefinite [7].

A vector r in T_pM is isotropic with respect to f if

$$f(r, r) = 0. \tag{4}$$

In every T_pM, for (M, g, Q), there exists an orthonormal Q-basis $\{i, Qi, Q^2i\}$ ([6]). From (1), (3) and (4) we state the following

Lemma 2.1. *Let $\{i, Qi, Q^2i\}$ be an orthonormal Q-basis of T_pM. If $r = ui + vQi + qQ^2i$ is an isotropic vector, then its coordinates satisfy*

$$uv + vq + qu = 0. \tag{5}$$

An isotropic (null) curves $c : r = r(t)$ are those whose tangent vectors are everywhere isotropic, i.e.,

$$f(dr, dr) = 0. \tag{6}$$

The physical forces are represented by vector fields. We investigate physical forces whose vectors are in T_pM on (M, g, Q). Moreover, these vectors are isotropic and they act along isotropic curves. We study the physical work done by such forces.

3. The work in T_pM

We consider an orthonormal Q-basis $\{i, Qi, Q^2i\}$ in T_pM on (M, g, Q).

Let p_{xyz} be a coordinate system such that the vectors i, Qi and Q^2i are on the axes p_x, p_y and p_z, respectively. So p_{xyz} is an orthonormal coordinate system.

The curve c is determined by

$$c : r(t) = x(t)i + y(t)Qi + z(t)Q^2i, \tag{7}$$

where $t \in [\alpha, \beta] \subset \mathbb{R}$.

Let c be an isotropic smooth curve. Thus equalities (1), (3), (6) and (7) imply

$$dxdy + dydz + dxdz = 0. \tag{8}$$

We determine a vector force field

$$F(x, y, z) = P(x, y, z)i + R(x, y, z)Qi + S(x, y, z)Q^2i, \tag{9}$$

where $P = P(x, y, z)$, $R = R(x, y, z)$, $S = S(x, y, z)$ are smooth functions.

Let the vector field F be isotropic. Hence following (5) we get

$$PR + RS + SP = 0. \tag{10}$$

Work A done by a force F, with respect to f, moving along a curve c is given by

$$A = \int_c f(F, dr), \tag{11}$$

where

$$dr = dxi + dyQi + dzQ^2i. \tag{12}$$

Case (A) Let F and c are both isotropic and they are on the same direction. Since c is a trajectory of F we have that the vectors F and dr are collinear. Therefore their coordinates satisfy

$$\frac{dx}{P} = \frac{dy}{R} = \frac{dz}{S} = \frac{1}{k}, \tag{13}$$

where $k \neq 0$ is a function. From (9) and (13) it follows $F = kdr$. Then, having in mind (6) and (11), we get $dA = f(kdr, dr) = kf(dr, dr) = 0$, i.e., $A = 0$.

Case (B) Now, we consider the case when F and c are both isotropic but they are on different directions.

From (3), (11) and (12) it follows

$$A = \int_c \big[P(dy + dz) + R(dx + dz) + S(dx + dy)\big], \tag{14}$$

and hence

$$A = \int_\alpha^\beta \big[P(y' + z') + R(x' + z') + S(x' + y')\big]dt. \tag{15}$$

- Let $dx + dy = 0$. From (8) we have $dx = dy = 0$ and $dz \neq 0$. Then $dr = dzQ^2i$. Therefore, using (15), we get

$$A = \int_\alpha^\beta \big(P(k_1, k_2, t) + R(k_1, k_2, t)\big)dt, \tag{16}$$

  where k_1 and k_2 are specific constants.

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- Let $P+R=0$. From (10) we have $P=R=0$ and hence $S\neq 0$. In this case equalities (11) and (15) imply
$$A=\int_{\alpha}^{\beta} S(x(t),y(t),z(t))[x'(t)+y'(t)]dt. \tag{17}$$
- Let $dx+dy\neq 0$ and $P+R\neq 0$. With the help of (8) and (10) we get
$$dz=-\frac{dxdy}{dx+dy}, \quad S=-\frac{PR}{P+R}. \tag{18}$$
We use (14), (15) and (18) and obtain that the work A is determined by
$$A=\int_{\alpha}^{\beta}\frac{(Py'-Rx')^2}{(P+R)(x'+y')}dt. \tag{19}$$

From Case (A) and Case (B) we state the following

Theorem 3.1. *Let f be the associated metric on (M,g,Q). Let p_{xyz} be a coordinate system such that the vectors i, Qi and Q^2i of the orthonormal Q-basis in T_pM are on the axes p_x, p_y and p_z, respectively. Let F be an isotropic vector force field moving along an isotropic curve c. Let A be the work done by F. Then*

(i) A is zero if F and c are on the same direction;

(ii) A is (16) if F and c are on different directions and $dx+dy=0$;

(iii) A is (17) if F and c are on different directions and $P+Q=0$;

(iv) A is (19) if F and c are on different directions and $dx+dy\neq 0$ and $P+R\neq 0$.

4. Work in a 2-plane

Now we consider an arbitrary 2-plane $\alpha=\{i,Qi\}$ in T_pM. We suppose that the angle $\varphi=\angle(i,Qi)$ belongs to the interval $(0,\frac{2\pi}{3}]$. On α we construct a coordinate system p_{xy} such that i is on the axis p_x and j is on the axis p_y, where
$$j=\frac{1}{\sin\varphi}(-\cos\varphi i+Qi). \tag{20}$$
We assume that $\|i\|=1$ and then p_{xy} is an orthonormal coordinate system.

In [5] it is proved the following

Theorem 4.1. *Let f be the associated metric on (M,g,Q) and let $\alpha=\{i,Qi\}$ be an arbitrary 2-plane in T_pM. Let the vector j be defined by (20) and p_{xy} be a coordinate system such that $i\in p_x$, $j\in p_y$. Then the equation of the circle $c:\ f(w,w)=a^2$ in α is given by*
$$(\cos\varphi)x^2+\frac{(1-\cos\varphi)(1+2\cos\varphi)}{\sin\varphi}xy-\frac{\cos^2\varphi}{1+\cos\varphi}y^2=\frac{a^2}{2}\ , \tag{21}$$
where $\varphi\in(0,\frac{2\pi}{3}]$.

Let $w=ui+vj$ be an isotropic vector, i.e., $f(w,w)=0$. Therefore, with the help of (21), we obtain
$$\cos^2\varphi\Big(\frac{y}{x}\Big)^2-\sin\varphi(1+2\cos\varphi)\frac{y}{x}-(1+\cos\varphi)\cos\varphi=0. \tag{22}$$
The discriminant of (22) is
$$D=(1+\cos\varphi)(1+3\cos\varphi).$$

Then we get the following cases:

Case (A) If $\varphi \in (\arccos(-\frac{1}{3}), \frac{2\pi}{3})$, then $D < 0$. There is no isotropic directions in T_pM.

Case (B) If $\varphi = \arccos(-\frac{1}{3})$, then $D = 0$. We have one isotropic straight line $c: \ y = \sqrt{2}x$. Then the force F and the curve c both are on one isotropic direction and the work A of the force F along c is zero.

Case (C) If $\varphi \in \left(0, \frac{\pi}{2}\right) \bigcup \left(\frac{\pi}{2}, \arccos(-\frac{1}{3})\right)$, then $D > 0$. We have two isotropic directions which generate two straight lines:

$$c_1 : y = k_1 x, \quad c_2 : y = k_2 x, \quad x \in [\alpha, \beta],$$

where k_1 and k_2 are solutions of the equation (22) for $\frac{y}{x}$.

- If F is on c_1, then the work of F along c_1 is zero. Similarly, if F is on c_2, then the work of F along c_2 is zero.
- We suppose that F is on c_2 but F acts on c_1. Then

$$F(x,y) = P(x,y)(i + k_2 j), \quad dr = dt(i + k_1 j). \tag{23}$$

Bearing in mind (2) and (20) we calculate

$$\begin{aligned} g(i,Qi) = g(Qi,i) = \cos\varphi, \quad g(i,Qj) = g(Qj,i) = \tfrac{\cos\varphi - \cos^2\varphi}{\sin\varphi}, \\ g(j,Qi) = g(Qi,j) = \sin\varphi, \quad g(j,Qj) = g(Qj,j) = -\tfrac{\cos^2\varphi}{1+\cos\varphi}. \end{aligned} \tag{24}$$

On the other hand the solutions k_1 and k_2 of (22) satisfy equalities

$$k_1 + k_2 = \frac{\sin\varphi(1 + 2\cos\varphi)}{\cos^2\varphi}, \quad k_1 k_2 = -\frac{1+\cos\varphi}{\cos\varphi}. \tag{25}$$

Using (3), (11), (23), (24) and (25) we find

$$A = \frac{1 + 3\cos\varphi}{\cos^2\varphi} \int_\alpha^\beta P(t, k_1 t)dt. \tag{26}$$

- Similarly, if F is on c_1 and F acts on c_2 we get

$$A = \frac{1 + 3\cos\varphi}{\cos^2\varphi} \int_\alpha^\beta P(t, k_2 t)dt.$$

Case (D) Finally, the condition $\varphi = \frac{\pi}{2}$ applied to (20) yields $j = Qi$. Then i and j are isotropic vectors. Therefore, from (3) and (11) it follows:

- $F = P(t,0)Qi$, $dr = (dt)i$. The work is $A = \int_\alpha^\beta P(t,0)dt$.
- $F = P(0,t)i$, $dr = (dt)Qi$. The work is $A = \int_\alpha^\beta P(0,t)dt$.

The results in Case (A) – Case (D) are summarized in Table 1.

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Table 1. Work $\boldsymbol{A}$ done by an isotropic vector force field F along an isotropic curve

φ	F acts on	trajectory of F	A
$(\arccos(-\frac{1}{3}), \frac{2\pi}{3})$	-	no is. curves	-
$\arccos(-\frac{1}{3})$	$c:\ y=\sqrt{2}x$	$c:\ y=\sqrt{2}x$	0
$\big(0,\frac{\pi}{2}\big)\bigcup\big(\frac{\pi}{2},\arccos(-\frac{1}{3})\big)$	$c_1: y=k_1x$	$c_1: y=k_1x$	0
$\big(0,\frac{\pi}{2}\big)\bigcup\big(\frac{\pi}{2},\arccos(-\frac{1}{3})\big)$	$c_2: y=k_2x$	$c_2: y=k_2x$	0
$\big(0,\frac{\pi}{2}\big)\bigcup\big(\frac{\pi}{2},\arccos(-\frac{1}{3})\big)$	$c_1: y=k_1x$	$c_2: y=k_2x$	$A=\frac{1+3\cos\varphi}{\cos^2\varphi}\int_\alpha^\beta P(t,k_1t)dt$
$\big(0,\frac{\pi}{2}\big)\bigcup\big(\frac{\pi}{2},\arccos(-\frac{1}{3})\big)$	$c_2: y=k_2x$	$c_1: y=k_1x$	$A=\frac{1+3\cos\varphi}{\cos^2\varphi}\int_\alpha^\beta P(t,k_2t)dt$
$\frac{\pi}{2}$	$c_1: x=0$	$c_2: y=0$	$A=\int_\alpha^\beta P(t,0)dt$
$\frac{\pi}{2}$	$c_1: y=0$	$c_2: x=0$	$A=\int_\alpha^\beta P(0,t)dt.$

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