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CURVATURE PROPERTIES OF RIEMANNIAN MANIFOLDS WITH CIRCULANT STRUCTURES

DIMITAR RAZPOPOV[1] AND GEORGI DZHELEPOV

ABSTRACT. We study a Riemannian manifold M equipped with a circulant structure Q, which is an isometry with respect to the metric. We consider two types of such manifolds: a 3-dimensional manifold M where the third power of Q is the identity, and a 4-dimensional manifold M where the fourth power of Q is the identity. In a single tangent space of M we have a special tetrahedron constructed by vectors of a Q-basis. The aim of the present paper is to find relations among the sectional curvatures of the 2-planes associated with the four faces of this tetrahedron and its cross sections passing through the medians and the edges of these faces.

1. INTRODUCTION

The study of pseudo-Riemannian manifolds with additional structures in differential geometry is of great interest to many mathematicians. Substantial results are associated with the sectional curvatures of some characteristic 2-planes, determined in every tangent space of the manifold (for instance [1], [5], [7], [8], [9], [11]).

In the present paper, we continue our research on the manifolds with additional structures, introduced in [6] and [10]. We consider a Riemannian manifold M equipped with a circulant structure Q, which is an isometry

[1] *corresponding author*

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with respect to the metric g. We study two classes of such manifolds determined by special properties of the curvature tensor. We find expressions of the curvatures of special 2-planes formed by vectors in a tangent space T_pM, $p \in M$.

First, we consider a 3-dimensional manifold (M, g, Q) where the third power of Q is the identity. In a single tangent space of (M, g, Q) we have a special tetrahedron constructed by vectors of a Q-basis of T_pM. We find a relation among the sectional curvatures generated by the four faces of the tetrahedron and its cross sections passing through the medians and the edges of these faces. Farther, we consider a 4-dimensional manifold (M, g, Q) where the fourth power of Q is the identity. We find a relation among the sectional curvatures of the faces and some cross sections of a tetrahedron constructed by vectors of a Q-basis of T_pM. Let us note that the obtained results for (M, g, Q) in the case when $n = 4$ are not similar to the results at $n = 3$.

2. Preliminaries

We consider a n-dimensional Riemannian manifold M with a metric g, equipped with an endomorphism Q in T_pM, such that $Q^n = \mathrm{id},\ Q \neq \pm\mathrm{id}$. Moreover, we suppose that Q is a circulant structure, i.e. the matrix of the components of Q in some basis is circulant. We assume that g is positive definite metric and Q is compatible with g such that

$$g(Qx, Qy) = g(x, y). \tag{2.1}$$

Here and anywhere in this work x, y, z, u will stand for arbitrary elements of the algebra of the smooth vector fields on M or vectors in T_pM.

We denote by (M, g, Q) the manifold M equipped with the metric g and the structure Q.

Let ∇ be the Riemannian connection of the metric g on (M, g, Q). The curvature tensor R of ∇ is determined by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

The tensor of type $(0, 4)$ associated with R is defined by the identity

$$R(x, y, z, u) = g(R(x, y)z, u).$$

We say that a manifold (M, g, Q) is in class $\mathcal{L}_0$ if the structure Q is parallel with respect to g, i.e., $\nabla Q = 0$.

We say that a manifold (M, g, Q) is in class $\mathcal{L}_1$ if

$$R(x, y, Qz, Qu) = R(x, y, z, u). \tag{2.2}$$

We say that a manifold (M, g, Q) is in class $\mathcal{L}_2$ if

$$R(Qx, Qy, Qz, Qu) = R(x, y, z, u). \tag{2.3}$$

In [3] and [10] it is shown that $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2$ are valid for cases $n = 3$ and $n = 4$ respectively.

Further, we will find the sectional curvatures of special 2-planes of T_pM when (M, g, Q) is a 3-dimensional manifold and also when (M, g, Q) is a 4-dimensional manifold. For this purpose, bearing in mind the well-known linear properties of the curvature tensor R, we obtain the following identity

$$\begin{aligned} &R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = \\ &R(Qx, Q^2x, Qx, Q^2x) + R(x, Qx, x, Qx) + R(x, Q^2x, x, Q^2x) \\ &+ 2R(x, Qx, Qx, Q^2x) - 2R(x, Q^2x, Qx, Q^2x) - 2R(x, Qx, x, Q^2x). \end{aligned} \tag{2.4}$$

If $\{x, y\}$ is a non-degenerate 2-plane spanned by vectors $x, y \in T_pM$, then its sectional curvature is

$$\mu(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g^2(x, y)} \ . \tag{2.5}$$

3. Curvature properties of a 3-dimensional (M, g, Q)

First, we recall facts from [4] and [6], which are necessary for our consideration.

Let (M, g, Q) be a 3-dimensional Riemannian manifold and let the local coordinates of Q be given by the circulant matrix

$$(Q_i^j) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence Q satisfies

$$Q^3 = \mathrm{id}. \tag{3.1}$$

We suppose that g is a positive definite metric and the property (2.1) holds.

A basis of type $\{x, Qx, Q^2x\}$ of T_pM is called a *Q-basis*. In this case we say that the vector x induces a Q-basis of T_pM.

The angles between the vectors of a Q-basis are

$$\angle(x, Qx) = \angle(Qx, Q^2x) = \angle(x, Q^2x) = \varphi, \tag{3.2}$$

where $\varphi \in (0, \frac{2\pi}{3})$. Evidently, an orthogonal Q-basis exists ([6]).

Theorem 3.1. [4] *Let (M, g, Q) satisfy (2.3). If a vector x induces a Q-basis, then for the sectional curvatures of the basic 2-planes we have*

$$\mu(x, Qx) = \mu(x, Q^2x) = \mu(Qx, Q^2x).$$

Due to Theorem 3.1, $(M, g, Q) \in \mathcal{L}_2$ has only one basic sectional curvature $\mu(x, Qx)$. It depends only on $\varphi = \angle(x, Qx)$ and we denote it by $\mu(\varphi)$ (see [4]).

Further in this section, we consider a tetrahedron whose faces are constructed by the 2-planes $\{x, Qx\}$, $\{Qx, Q^2x\}$ and $\{x, Q^2x\}$. The base of the tetrahedron is constructed by the 2-plane $\alpha = \{Qx - x, Qx - Q^2x\}$.

Without loss of generality we suppose that $g(x, x) = 1$. Hence, using (2.1) and (3.2), we calculate

$$\begin{aligned} g(x, Qx) = \cos\varphi, \qquad g(Qx - x, Qx - Q^2x) = 1 - \cos\varphi, \\ g(Qx - x, Qx - x) = g(Qx - Q^2x, Qx - Q^2x) = 2 - 2\cos\varphi, \end{aligned} \tag{3.3}$$

which implies that the base of the terahedron is an equilateral triangle.

In the next theorem, we obtain an expression of the sectional curvature of α by the sectional curvature of $\{x, Qx\}$ and by the sectional curvature of $\beta = \{Q^2x, Qx + x\}$. The 2-plane β determines a cross section of the tetrahedron.

Theorem 3.2. *Let (M, g, Q) belong to $\mathcal{L}_2$. Then the curvature of the 2-plane $\alpha = \{Qx - x, Qx - Q^2x\}$ is*

$$\mu(\alpha) = \frac{3(1 + \cos\varphi)}{1 - \cos\varphi}\mu(\varphi) - \frac{2(1 + 2\cos\varphi)}{1 - \cos\varphi}\mu(\beta), \tag{3.4}$$

where $\varphi = \angle(x, Qx)$, $\beta = \{Q^2x, Qx + x\}$.

Proof. The conditions (2.3) and (3.1) imply ([3]):

$$R_1 = R(x, Qx, x, Qx) = R(x, Q^2x, x, Q^2x) = R(Qx, Q^2x, Qx, Q^2x), \tag{3.5}$$

$$R_2 = R(x, Qx, x, Q^2x) = R(x, Q^2x, Qx, Q^2x) = R(x, Qx, Q^2x, Qx). \tag{3.6}$$

Then from (2.4) we get

$$R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = 3R_1 - 6R_2. \tag{3.7}$$

On the other hand, taking into account (3.5) and (3.6), we calculate

$$R(Q^2x, Qx + x, Q^2x, Qx + x) = 2(R_1 + R_2). \tag{3.8}$$

Together with (3.7), (3.8) yields

$$\begin{aligned} R(Qx - x, Qx - Q^2x, &Qx - x, Qx - Q^2x) \\ &= 9R_1 - 3R(Q^2x, Qx + x, Q^2x, Qx + x). \end{aligned} \tag{3.9}$$

Now, using (2.1) and (3.2), we find

$$g(Qx + x, Qx + x) = 2 + 2\cos\varphi, \quad g(Q^2x, Qx + x) = 2\cos\varphi. \tag{3.10}$$

We apply equalities (3.3), (3.9) and (3.10) in (2.5), and obtain (3.4). □

In our previous work, we obtain the following relation among the sectional curvatures of 2-planes of the type $\{x, Qx\}$, whose basis vectors x and Qx determine angles φ, $\frac{\pi}{2}$ and $\frac{\pi}{3}$, respectively.

Theorem 3.3. [4] *Let (M, g, Q) satisfy (2.3). If a vector x induces a Q-basis, then*

$$\mu(\varphi) = \frac{1 - 2\cos\varphi}{1 + \cos\varphi}\mu(\frac{\pi}{2}) + \frac{3\cos\varphi}{1 + \cos\varphi}\mu(\frac{\pi}{3}),$$

where $\varphi = \angle(x, Qx)$.

From Theorem 3.2 and Theorem 3.3 we establish the following

Proposition 3.1. *Let (M, g, Q) belong to $\mathcal{L}_2$. Then the curvatures of the 2-planes $\alpha = \{Qx - x, Qx - Q^2x\}$ and $\beta = \{Q^2x, x + Qx\}$ satisfy*

$$\mu(\alpha) = \frac{3}{1 - \cos\varphi}\big((1 - 2\cos\varphi)\mu(\frac{\pi}{2}) + 3\cos\varphi\mu(\frac{\pi}{3})\big) - \frac{2(1 + 2\cos\varphi)}{1 - \cos\varphi}\mu(\beta).$$

Corollary 3.1. *If (M, g, Q) belongs to $\mathcal{L}_2$ and $\varphi = \frac{\pi}{2}$, then*

$$\mu(\alpha) = 3\mu(\frac{\pi}{2}) - 2\mu(\beta).$$

Further, for a manifold $(M, g, Q) \in \mathcal{L}_1$ we find an expression of $\mu(\alpha)$ by $\mu(\varphi)$. Also we get $\mu(\beta)$.

Theorem 3.4. *Let (M, g, Q) belong to $\mathcal{L}_1$. Then the curvatures of the 2-planes $\alpha = \{Qx - x, Qx - Q^2x\}$ and $\beta = \{Q^2x, x + Qx\}$ are*

$$\mu(\alpha) = 3\cot^2\frac{\varphi}{2}\mu(\varphi), \quad \mu(\beta) = 0, \tag{3.11}$$

where $\varphi = \angle(x, Qx)$.

Proof. From (2.2), (3.5) and (3.6) we get $R_1 = -R_2$. Thus, equalities (3.7) and (3.8) become

$$\begin{aligned} &R(Qx - x, Qx - Q^2x, Qx - x, Qx - Q^2x) = 9R_1, \\ &R(Q^2x, Qx + x, Q^2x, Qx + x) = 0. \end{aligned} \tag{3.12}$$

Now, applying (3.3), (3.10) and (3.12) in (2.5), we obtain (3.11). □

Corollary 3.2. *Let (M, g, Q) belong to $\mathcal{L}_1$. Then*

i) *the inequality $\mu(\alpha) > \mu(\varphi)$ holds;*
ii) $\mu(\alpha) = 3\mu(\frac{\pi}{2})$.

Proof. i) The cotangent function is decreasing in the interval $(0, \pi)$. Therefore, bearing in mind the condition $\varphi \in (0, \frac{2\pi}{3})$, we get $\cot\frac{\varphi}{2} > \frac{\sqrt{3}}{3}$. Hence, because of the first equality of (3.11), we have that $\mu(\alpha) > \mu(\varphi)$ for every $\varphi \in (0, \frac{2\pi}{3})$.

ii) If we put $\varphi = \frac{\pi}{2}$ into (3.11), then the proof follows. □

4. Curvature properties of a 4-dimensional (M, g, Q)

In the beginning of this section we recall some basic facts for a 4-dimensional (M, g, Q), known from [2] and [10].

Let (M, g, Q) be a 4-dimensional Riemannian manifold and let the local coordinates of Q be given by the circulant matrix

$$(Q_i^j) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence Q satisfies

$$Q^4 = \mathrm{id}, \qquad Q^2 \neq \pm\mathrm{id}. \tag{4.1}$$

We assume that g is a positive definite metric and the property (2.1) is valid.

A basis of type $\{x, Qx, Q^2x, Q^3x\}$ of T_pM is called a Q-*basis*. In this case we say that the vector x induces a Q-basis of T_pM. The angles between the vectors of a Q-basis are as follows

$$\begin{aligned}&\angle(x, Qx) = \angle(Qx, Q^2x) = \angle(x, Q^3x) = \angle(Q^2x, Q^3x) = \varphi,\\ &\angle(x, Q^2x) = \angle(Qx, Q^3x) = \theta,\end{aligned} \tag{4.2}$$

where $\varphi \in (0, \pi)$, $\theta \in (0, \pi)$.

In [10], it is proved the inequality $3-4\cos\varphi+\cos\theta < 0$ and the existence of an orthogonal Q-basis.

Theorem 4.1. [2] *Let (M, g, Q) belong to $\mathcal{L}_2$. If a vector x induces a Q-basis, then for the sectional curvatures of the basic 2-planes we have*

$$\begin{aligned}&\mu(x, Qx) = \mu(Qx, Q^2x) = \mu(Q^2x, Q^3x) = \mu(Q^3x, x),\\ &\mu(x, Q^2x) = \mu(Qx, Q^3x).\end{aligned}$$

Due to Theorem 4.1 there are only two basic sectional curvatures. They are $\mu(x, Qx)$ and $\mu(x, Q^2x)$. The sectional curvature $\mu(x, Qx)$ depends on $\varphi = \angle(x, Qx)$. We denote $\mu(\varphi) = \mu(x, Qx)$.

Let x induce a Q-basis of T_pM. Then the vectors x, Qx and Q^2x determine a tetrahedron, whose faces are constructed by the 2-planes $\{x, Qx\}$, $\{Qx, Q^2x\}$ and $\{x, Q^2x\}$. The base of the tetrahedron is constructed by the 2-plane $\alpha = \{Qx - x, Qx - Q^2x\}$.

Without loss of generality we suppose $g(x, x) = 1$. Thus, it follows from (2.1) and (4.2) that:

$$\begin{aligned}&g(x, Qx) = \cos\varphi, \quad g(x, Q^2x) = \cos\theta,\\ &g(Qx - x, Qx - Q^2x) = 1 - 2\cos\varphi + \cos\theta,\\ &g(Qx - x, Qx - x) = g(Qx - Q^2x, Qx - Q^2x) = 2 - 2\cos\varphi\,.\end{aligned} \tag{4.3}$$

The latter equalities show that the base of the tetrahedron is an isosceles triangle. In the following theorems, we obtain an expression of the sectional curvature of α by the sectional curvatures of $\{x, Qx\}$ and $\{x, Q^2x\}$, and also by the sectional curvatures of $\gamma = \{Q^2x, x + Qx\}$, $\delta = \{x, Qx + Q^2x\}$ and $\sigma = \{Qx, x + Q^2x\}$. The 2-planes γ, δ and σ determine cross sections of the tetrahedron.

Theorem 4.2. *Let (M, g, Q) belong to $\mathcal{L}_2$. Then the curvature of the 2-plane $\alpha = \{Qx - x, Qx - Q^2x\}$ is*

$$\begin{aligned}\mu(\alpha) = &\frac{1}{(1-\cos\theta)(3-4\cos\varphi+\cos\theta)}\Big(6(1-\cos^2\varphi)\mu(\varphi)\\ &+3(1-\cos^2\theta)\mu(\beta)-2(1+\cos\theta-2\cos^2\varphi)\mu(\sigma)\\ &-\big(2+2\cos\varphi-(\cos\varphi+\cos\theta)^2\big)\big(\mu(\gamma)+\mu(\delta)\big)\Big),\end{aligned} \tag{4.4}$$

where $\beta = \{x, Q^2x\}$, $\gamma = \{Q^2x, Qx + x\}$, $\delta = \{x, Qx + Q^2x\}$, $\sigma = \{Qx, x + Q^2x\}$.

Proof. We denote

$$\begin{aligned}&R_1 = R(x, Qx, x, Qx), \quad R_2 = R(x, Q^2x, x, Q^2x),\\ &R_3 = R(x, Qx, Q^2x, x), \quad R_4 = R(x, Qx, Qx, Q^2x),\\ &R_5 = R(x, Q^2x, Q^2x, Qx).\end{aligned} \tag{4.5}$$

Then, from (2.3), (2.4) and (4.1), we get

$$R(Qx-x, Qx-Q^2x, Qx-x, Qx-Q^2x) = 2R_1+2R_3+R_2+2R_4+2R_5, \tag{4.6}$$

On the other hand, using (4.5), we calculate

$$\begin{aligned}&R(Q^2x, Qx+x, Q^2x, Qx+x) = R_1+R_2-2R_5,\\ &R(x, Qx+Q^2x, x, Qx+Q^2x) = R_1+R_2-2R_3,\\ &R(Qx, x+Q^2x, Qx, x+Q^2x) = 2R_1-2R_4.\end{aligned} \tag{4.7}$$

Applying (4.7) in (4.6) we find

$$\begin{aligned}R(Qx-x, Qx-Q^2x, Qx-x, Qx-Q^2x) = 6R_1+3R_2&\\ -R(Q^2x, Qx+x, Q^2x, Qx+x)&\\ -R(x, Qx+Q^2x, x, Qx+Q^2x)&\\ -R(Qx, x+Q^2x, Qx, x+Q^2x)).&\end{aligned} \tag{4.8}$$

From (2.1), (4.2) and (4.3) we have

$$\begin{aligned}&g(Qx+x, Qx+x) = g(Qx+Q^2x, Qx+Q^2x) = 2+2\cos\varphi,\\ &g(x, Qx+Q^2x) = g(Q^2x, Qx+x) = \cos\theta+\cos\varphi,\\ &g(x+Q^2x, x+Q^2x) = 2+2\cos\theta, \quad g(Qx, x+Q^2x) = 2\cos\varphi.\end{aligned} \tag{4.9}$$

Therefore, (2.5), (4.3), (4.8) and (4.9) imply (4.4). □

Corollary 4.1. *Let (M,g,Q) belong to $\mathcal{L}_2$. If $\varphi=\theta$, then*

$$\mu(\alpha)=\frac{1}{3(1-\cos\varphi)}\Big((1+\cos\varphi)\big(6\mu(\varphi)+3\mu(\beta)\big)\\ -2(1+2\cos\varphi)\big(\mu(\gamma)+\mu(\delta)+\mu(\sigma)\big)\Big).$$

In particular, if $\varphi=\theta=\frac{\pi}{2}$, then

$$\mu(\alpha)=\frac{1}{3}\Big(6\mu(\frac{\pi}{2})+3\mu(\beta)-2\mu(\gamma)-2\mu(\delta)-2\mu(\sigma)\Big).$$

Now, for a manifold $(M,g,Q)\in\mathcal{L}_1$ we find expressions of $\mu(\alpha)$, $\mu(\beta)$, $\mu(\sigma)$, $\mu(\gamma)$ and $\mu(\delta)$ by φ, θ and $\mu(\varphi)$.

Theorem 4.3. *Let (M,g,Q) belong to $\mathcal{L}_1$. Then the curvatures of the 2-planes $\alpha=\{Qx-x,Qx-Q^2x\}$, $\beta=\{x,Q^2x\}$, $\gamma=\{Q^2x,Qx+x\}$, $\delta=\{x,Qx+Q^2x\}$ and $\sigma=\{Qx,x+Q^2x\}$ are*

$$\mu(\alpha)=\frac{4(1+\cos\varphi)}{3-4\cos\varphi+\cos\theta}\mu(\varphi),\quad \mu(\beta)=\mu(\sigma)=0,\\ \mu(\gamma)=\mu(\delta)=\frac{1-\cos^2\varphi}{2+2\cos\varphi+(\cos\varphi-\cos\theta)^2}\mu(\varphi). \tag{4.10}$$

Proof. By using (2.2) and (4.5) we get that $R_1=R_4$ and $R_2=R_3=R_5=0$. Thus, from (4.6) and (4.7), we have

$$R(Qx-x,Qx-Q^2x,Qx-x,Qx-Q^2x)=4R_1,$$
$$R(Qx,x+Q^2x,Qx,x+Q^2x)=0,$$
$$R(x,Qx+Q^2x,x,Qx+Q^2x)=R(Q^2x,Qx+x,Q^2x,Qx+x)=R_1.$$

We apply the latter equalities, (4.3) and (4.9) in (2.5) and obtain (4.10). □

Finally, due to Theorem 4.3, we state the following

Corollary 4.2. *Let (M,g,Q) belong to $\mathcal{L}_1$. If $\varphi=\theta$, then*

$$\mu(\alpha)=\frac{4}{3}\cot^2\frac{\varphi}{2}\mu(\varphi),\qquad \mu(\gamma)=\mu(\delta)=\frac{1-\cos^2\varphi}{2+2\cos\varphi}\mu(\varphi).$$

In particular, if $\varphi=\theta=\frac{\pi}{2}$, then

$$\mu(\alpha)=\frac{4}{3}\mu(\frac{\pi}{2}),\qquad \mu(\gamma)=\mu(\delta)=\frac{1}{2}\mu(\frac{\pi}{2}).$$

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Department of Mathematics and Informatics
Agricultural University of Plovdiv
12 Mendeleev Blvd, 4000 Plovdiv, Bulgaria
E-mail address: razpopov@au-plovdiv.bg

Department of Mathematics and Informatics
Agricultural University of Plovdiv
12 Mendeleev Blvd, 4000 Plovdiv, Bulgaria
E-mail address: dzhelepov@au-plovdiv.bg