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On an Indefinite Metric on a Four-Dimensional Riemannian Manifold

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Abstract: Our research focuses on the tangent space of a point on a four-dimensional Riemannian manifold. Besides having a positive definite metric, the manifold is endowed with an additional tensor structure of type $(1, 1)$, whose fourth power is minus the identity. The additional structure is skew-circulant and compatible with the metric, such that an isometry is induced in every tangent space on the manifold. Both structures define an indefinite metric. With the help of the indefinite metric, we determine circles in different two-planes in the tangent space on the manifold. We also calculate the length and area of the circles. On a smooth closed curve, such as a circle, we define a vector force field. Further, we obtain the circulation of the vector force field along the curve, as well as the flux of the curl of this vector force field across the curve. Finally, we find a relation between these two values, which is an analog of the well-known Green's formula in the Euclidean space.

Keywords: Riemannian manifold; indefinite metric tensor; length; area; Green's formula

MSC: 53B30; 53A04; 26B15; 26B20

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1. Introduction

Riemannian manifolds with additional tensor structures are extensively studied in modern differential geometry. Riemannian almost product manifolds and almost Hermitian manifolds are examples of such manifolds. Four-dimensional Riemannian manifolds with circulant structures are associated with Riemannian almost product manifolds [1], and four-dimensional Riemannian manifolds with skew-circulant structures are associated with Hermitian manifolds [2].

In this paper, we continue our research, previously addressed in [2,3], on Riemannian manifolds with an additional tensor structure, whose fourth power is minus the identity. The additional structure is skew-circulant, i.e., its components form a skew-circulant matrix. The properties and some applications of such matrices can be found in [4–9]. We define an indefinite metric on the manifold using the Riemannian metric and the additional structure, and obtain some useful formulae with respect to this metric, which are analogs of well-known formulae (such as length and area of a circle, circulation of a vector force field, and Green's formula) in the Euclidean case. We can find more motivations for our work from several papers (see [10–12]).

If k is a simple closed curve in a plane, then it surrounds some region in the plane. Green's theorem transforms the line integral around k into a double integral over the region inside k. In physics, this provides the relationship between the circulation $C = \oint_k F.ds$ of the vector force field F around the path k and the flux, done by the curl of F, across the region inside k.

Green's theorem is a special case of Stokes' theorem. Both theorems are widely used in the study of electric and magnetic fields. The modern approach to these theorems on manifolds using differential forms is exhibited, for example, in [13–18].

We consider a four-dimensional Riemannian manifold M with an additional tensor field S of type $(1,1)$, whose fourth power is minus the identity. The structure S is compatible with the metric g, such that an isometry is induced in every tangent space T_pM on M. Both structures, g and S, define an indefinite metric $\tilde{g}$ [2]. The metric $\tilde{g}$ determines space-like, isotropic and time-like vectors in T_pM. We consider circles k_i, with respect to $\tilde{g}$, in special two-planes β_i of T_pM, constructed on space-like or time-like vectors.We calculate their lengths and areas (with respect to $\tilde{g}$), which, in some cases, are imaginary or negative numbers. We note that some problems related to circles, concerning their lengths or areas considered in terms of indefinite metrics, are addressed in [19–23]. Finally, we obtain analogs of Green's theorem that provide a relation between the circulation of the vector force field F around a closed curve (in particular, a circle) k_i in β_i and the flux, done by the curl of F, across the region inside k_i.

The paper is organized as follows. In Section 2, we provide some facts, definitions and statements, which are necessary for the present considerations. In Section 3, we introduce a special two-plane β_1 of T_pM and determine an equation of a circle k_1 in β_1 with respect to $\tilde{g}$. In Sections 3.1 and 3.2 we calculate the length and area of k_1. In Section 3.3, we find the circulation of a vector force field F around the smooth closed curve, k_1, and the flux, done by the curl of F, across the region inside k_1. In Section 4, we introduce a two-plane, β_2, of T_pM and determine an equation of a circle k_2 in β_2 with respect to $\tilde{g}$. Further, we calculate the length and area of k_2. We derive the circulation of a vector force field F around a smooth closed curve k_2 and the flux, done by the curl of F, across the region inside k_2. All values obtained in Sections 3 and 4 are calculated with respect to $\tilde{g}$.

2. Preliminaries

In this paper, we study a four-dimensional Riemannian manifold, equipped with tensor structures whose component matrices are right skew-circulant. Thus, we recall the definition of such matrices. They are Toeplitz matrices, and were addressed in [4,6].

The real right skew-circulant matrix with the first row $(a_1, a_2, a_3, a_4) \in \mathbf{R}^4$ is a square matrix of the form:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_4 & a_1 & a_2 & a_3 \\ -a_3 & -a_4 & a_1 & a_2 \\ -a_2 & -a_3 & -a_4 & a_1 \end{pmatrix}.$$

The skew-circulant matrices form a vector space with the following basis:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Further, we use the matrix $S = E_2$. We note that E is the identity matrix and $S^2 = E_3$, $S^3 = E_4$. These matrices are also a part of our considerations. The square of S gives $J = E_3$, which is an example of the well-known complex structure.

We equip a four-dimensional differentiable manifold M with a tensor structure S of type $(1,1)$, such that S satisfies

$$S^4 = -\mathrm{id}. \tag{1}$$

We suppose that, at each point on M, the component matrix of S, with respect to a basis in the tangent space T_pM, is skew-circulant.

The matrix E_2 is one solution of the equation $S^4 = -E$. Then, we consider the skew-circulant structure S whose component matrix, with respect to a basis in T_pM, is

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Obviously, S satisfies (1) and $J = S^2$ is a complex structure.

Let g be a positive definite metric on M, which is compatible with S, i.e.,

$$g(Su, Sv) = g(u, v). \tag{3}$$

Here, and anywhere in this work, u, v, w, e_1, e_2 stand for arbitrary smooth vector fields on M or arbitrary vectors in the tangent space T_pM, $p \in M$.

The conditions (2) and (3) imply that the matrix of g, with respect to the same basis, has the following form:

$$(g_{ij}) = \begin{pmatrix} A & B & 0 & -B \\ B & A & B & 0 \\ 0 & B & A & B \\ -B & 0 & B & A \end{pmatrix},$$

where $A = A(p)$ and $B = B(p)$ are smooth functions of an arbitrary point p on M. Moreover, $A(p) > \sqrt{2}B(p) > 0$ in order for g to be positive definite.

Such a manifold (M, g, S) was introduced in [2].

If u is a nonzero vector on (M, g, S), then, according to (2), we have $Su \neq \pm u$. Thus, the angle φ between u and Su belongs to the interval $(0, \pi)$. The vectors u, Su, S^2u and S^3u determine six angles, which satisfy equalities [2]:

$$\begin{gathered} \angle(u, Su) = \angle(Su, S^2u) = \angle(S^2u, S^3u) = \varphi, \\ \angle(S^3u, u) = \pi - \varphi, \ \angle(u, S^2u) = \angle(Su, S^3u) = \tfrac{\pi}{2}. \end{gathered} \tag{4}$$

Definition 1. *A basis of type $\{S^3u, S^2u, Su, u\}$ of T_pM is called an S-basis. In this case, we say that the vector u induces an S-basis of T_pM.*

In [2], the conditions under which such a basis exists are described, as well as the following statement:

Lemma 1. *Let a vector u induce an S-basis $\{S^3u, S^2u, Su, u\}$ in T_pM. Then, the angle $\varphi = \angle(u, Su)$ satisfies inequalities*

$$\frac{\pi}{4} < \varphi < \frac{3\pi}{4}. \tag{5}$$

The associated metric $\tilde{g}$ on (M, g, S) is determined by

$$\tilde{g}(u, v) = g(u, Sv) + g(Su, v). \tag{6}$$

The matrix of its components is

$$(\tilde{g}_{ij}) = \begin{pmatrix} 2B & A & 0 & -A \\ A & 2B & A & 0 \\ 0 & A & 2B & A \\ -A & 0 & A & 2B \end{pmatrix}.$$

Two of the eigenvalues of $\tilde{g}_{ij}$ are negative, and the other two are positive. So, $\tilde{g}$ has signature $(2,2)$ and it is an indefinite metric [2].

According to (6), for an arbitrary vector v the following is valid:

$$\tilde{g}(v,v) = 2g(v,Sv) = R^2,\ R^2 \in \mathbb{R}. \tag{7}$$

The norm of every vector u and the cosine of φ are given by the following equalities:

$$\|u\| = \sqrt{g(u,u)}, \quad \cos\varphi = \frac{g(u,Su)}{g(u,u)}. \tag{8}$$

In the rest of the paper, we assume that $\|u\| = 1$ and, using (8), we obtain

$$\cos\varphi = g(u,Su). \tag{9}$$

Due to (3), (6), (8) and (9) we state that the normalized S-basis $\{S^3u, S^2u, Su, u\}$ satisfies the following equalities:

$$\begin{aligned}
&\tilde{g}(u,u) = \tilde{g}(Su,Su) = \tilde{g}(S^2u,S^2u) = \tilde{g}(S^3u,S^3u) = 2\cos\varphi,\\
&\tilde{g}(u,Su) = \tilde{g}(Su,S^2u) = \tilde{g}(S^2u,S^3u) = -\tilde{g}(S^3u,u) = 1.\\
&\tilde{g}(u,S^2u) = \tilde{g}(Su,S^3u) = 0.
\end{aligned} \tag{10}$$

A circle k in a two-plane of T_pM of a radius R centered at the origin $p \in T_pM$, with respect to the associated metric $\tilde{g}$ on (M,g,S), is determined by (7), where v is the radius vector of an arbitrary point on k.

Further, we consider circles k_1 and k_2, and the regions D_1 and D_2 inside them, in two different subspaces β_1 and β_2 of T_pM, spanned by two-planes $\{u, S^2u\}$ and $\{u, Su\}$, respectively. According to (4), (5) and (10), the vectors u, Su, S^2u and S^3u are space-like if, and only if, $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, and they are time-like if, and only if, $\varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. Therefore, the two-planes β_1 and β_2 are space-like when $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, and they are time-like when $\varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$.

Remark 1. *The rest of the two-planes $\{u, S^3u\}$, $\{Su, S^2u\}$, $\{Su, S^3u\}$, $\{S^2u, S^3u\}$ constructed with the basis vectors of $\{S^3u, S^2u, Su, u\}$ in T_pM have the same properties as β_1 or β_2.*

We note that the two-plane β_1 also belongs to the tangent space of the associated Hermitian manifold $(M, g, J = S^2)$ with the complex structure J.

3. Circles in the Two-Plane β_1

Due to (4), it is true that the vectors u and S^2u form an orthonormal basis of β_1. The coordinate system p_{xy} on β_1, wherein u is on the axis p_x and S^2u is on the axis p_y, is an orthonormal coordinate system of β_1.

A circle k_1 in β_1 centered at the origin p, with respect to $\tilde{g}$ on (M,g,S), is defined by (7). The equation of k_1 with respect to p_{xy} is obtained as follows:

Theorem 1 ([3])**.** *Let $\tilde{g}$ be the associated metric on (M,g,S) and let β_1 be a two-plane in T_pM with a basis $\{u, S^2u\}$. If p_{xy} is a coordinate system such that $u \in p_x$, $S^2u \in p_y$, then, the equation of the circle (7) in β_1 is given by:*

$$2\cos\varphi x^2 + 2\cos\varphi y^2 = R^2. \tag{11}$$

The only closed curve k_1, determined by (11), is a circle in terms of g, with the following parameters:

Case (A) $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $R^2 > 0$;

Case (B) $\varphi \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ and $R^2 < 0$.

The two-plane β_1 is constructed on space-like vectors in Case (A), and β_1 is constructed on time-like vectors in Case (B).

3.1. Length of a Circle with Respect to $\tilde{g}$

Firstly, we consider Case (A). The circle (7) has a radius $R > 0$ and the angle φ satisfies

$$\frac{\pi}{4} < \varphi < \frac{\pi}{2}. \tag{12}$$

Theorem 2. *The circle k_1 with (12) and a radius $R > 0$ has length*

$$L = 2\pi R. \tag{13}$$

Proof. Let $v = xu + yS^2u$ be a radius vector of an arbitrary point on the circle k_1. Then, $\mathrm{d}v = \mathrm{d}xu + \mathrm{d}yS^2u$ is a tangent vector on k_1. The length L of k_1 with respect to $\tilde{g}$ is determined, as usual, by:

$$\mathrm{d}L = \sqrt{\tilde{g}(\mathrm{d}v, \mathrm{d}v)}.$$

Then, using (10) and (11), we obtain:

$$L = \oint_{k_1} \sqrt{2\cos\varphi \mathrm{d}x^2 + 2\cos\varphi \mathrm{d}y^2}. \tag{14}$$

We substitute

$$x = \frac{R}{\sqrt{2\cos\varphi}}\cos t,\ y = \frac{R}{\sqrt{2\cos\varphi}}\sin t,\ t \in [0, 2\pi],$$

into (14) and find (13). $\square$

Now, we consider Case (B). The circle k_1 has a radius $R = ri,\ r > 0,\ i^2 = -1$ and the angle φ satisfies

$$\frac{\pi}{2} < \varphi < \frac{3\pi}{4}. \tag{15}$$

Therefore, the Equation (11) transforms into

$$2\cos\varphi x^2 + 2\cos\varphi y^2 = -r^2. \tag{16}$$

By calculations similar to those of Case (A) we find the integral (14) over k_1 with (16). Using the substitutions

$$x = \frac{r}{\sqrt{-2\cos\varphi}}\cos t,\ y = \frac{r}{\sqrt{-2\cos\varphi}}\sin t,\ t \in [0, 2\pi],$$

we obtain

Proposition 1. *The circle k_1 with (15) and a radius $R = ri,\ r > 0,\ i^2 = -1$ has an imaginary length*

$$L = 2\pi R.$$

3.2. Area of a Circle with Respect to $\tilde{g}$

For Case (A) we state the following:

Theorem 3. *The area A'' of the circle k_1 with (12) with a radius $R > 0$ is*

$$A'' = \pi R^2. \tag{17}$$

Proof. We denote, by $\widetilde{\cos}\angle(u, S^2u)$ and $\widetilde{\sin}\angle(u, S^2u)$, the cosine and the sine of the angle $\angle(u, S^2u)$ with respect to $\tilde{g}$. Considering $\tilde{g}(u, S^2u) = 0$ (presented in (10)), we have

$$\widetilde{\cos}\angle(u, S^2u) = 0,$$

and, hence,

$$\widetilde{\sin}\angle(u, S^2u) = 1. \tag{18}$$

In the coordinate plane p_{xy}, we construct a parallelogram with locus vectors $\mathrm{d}xu$ and $\mathrm{d}ySu$. For its area A'' with respect to $\tilde{g}$ we obtain

$$\mathrm{d}A'' = \sqrt{\tilde{g}(\mathrm{d}xu, \mathrm{d}xu)}\sqrt{\tilde{g}(\mathrm{d}ySu, \mathrm{d}ySu)}\widetilde{\sin}\angle(u, S^2u).$$

We apply (10) and (18) in the latter equality and find

$$\mathrm{d}A'' = 2\cos\varphi\mathrm{d}x\mathrm{d}y. \tag{19}$$

We integrate (19) over the region D_1 inside k_1 and calculate

$$A'' = 2\cos\varphi\int\int_{D_1}\mathrm{d}x\mathrm{d}y, \tag{20}$$

with

$$D_1:\ 2\cos\varphi x^2 + 2\cos\varphi y^2 \leq R^2.$$

We substitute

$$x = \frac{R}{\sqrt{2\cos\varphi}}\rho\cos t,\ y = \frac{R}{\sqrt{2\cos\varphi}}\rho\sin t,\ t \in [0, 2\pi],\ \rho \in [0, 1],$$

and Jacobian $\triangle = \frac{R^2}{2\cos\varphi}\rho$ into the integral (20) and obtain (17). □

Now, we consider Case (B).

Proposition 2. *The area A'' of the circle k_1 with (15) and a radius $R = ri,\ r > 0,\ i^2 = -1$ has a negative value*

$$A'' = \pi R^2. \tag{21}$$

Proof. The circle k_1 has an Equation (16) with conditions (15) and a radius $R = ri$, where $r > 0$, $i^2 = -1$. By calculations similar to those of Case (A), we find that the area of k_1 is given by

$$A'' = 2\cos\varphi\int\int_{D_1}\mathrm{d}x\mathrm{d}y, \tag{22}$$

with

$$D_1:\ -2\cos\varphi x^2 - 2\cos\varphi y^2 \leq r^2.$$

We substitute

$$x = \frac{r}{\sqrt{-2\cos\varphi}}\rho\cos t,\ y = \frac{r}{\sqrt{-2\cos\varphi}}\rho\sin t,\ t \in [0, 2\pi],\ \rho \in [0, 1],$$

and Jacobian $\triangle = \frac{r^2}{2\cos\varphi}\rho$ into the integral (22) and obtain $A'' = -\pi r^2$, which implies (21). □

3.3. Circulation and Flux with Respect to $\tilde{g}$

We consider a closed curve k_1 in β_1, given by

$$x = x(t), \quad y = y(t), \quad t \in [t_1, t_2], \tag{23}$$

where $x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$.

Let
$$F(x,y) = P(x,y)u + Q(x,y)S^2u \tag{24}$$
be a vector force field on the curve k_1.

For the circulation C of a vector field F along a curve k we assume the following definition:
$$C = \oint_k \tilde{g}(F, \mathrm{d}v), \tag{25}$$
where v is the radius vector of a point on k.

We denote, by D_1, the region inside k_1. For both cases (A) and (B) of circle (11) the following statements hold.

Theorem 4. *The circulation C, done by the force (24) along the curve (23), is expressed by*
$$C = 2\cos\varphi \oint_{k_1} (P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y), \tag{26}$$
where $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$.

Proof. Let $v = xu + yS^2u$ be the radius vector of a point on k_1. By virtue of (10) and (24), and bearing in mind $\mathrm{d}v = \mathrm{d}xu + \mathrm{d}yS^2u$, we obtain
$$\tilde{g}(F, \mathrm{d}v) = 2\cos\varphi(P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y). \tag{27}$$

Obviously, (26) follows from (23), (25) and (27). $\square$

We determine a vector w in T_pM by the equality
$$w = \frac{1}{\sqrt{1 - 2\cos^2\varphi}}\Big(\cos\varphi u - Su + \cos\varphi S^2u\Big), \tag{28}$$
where $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. By using (1), (3) and (9) it is easy to verify that
$$g(w,u) = g(w, S^2u) = 0, \quad g(w,w) = 1.$$

We construct an orthonormal coordinate system $Oxyz$, such that $u \in Ox$, $S^2u \in Oy$, $w \in Oz$.

We suppose that the curl of F, determined by (24), with respect to $Oxyz$, is
$$\mathrm{curl}\, F = (Q_x - P_y)w.$$

The flux T of the vector field curl F across the region D_1 inside the curve k_1 is given by
$$T = \int\int_{D_1} \tilde{g}(\mathrm{curl}\, F, w)\mathrm{d}A''. \tag{29}$$

With the help of (10) and (28) we obtain $\tilde{g}(w,w) = -2\cos\varphi$. Then, from (19) and (29) we state the following.

Theorem 5. *The flux T of the vector field curl F across the region D_1 inside the curve (23) is expressed by*
$$T = -4\cos^2\varphi \int\int_{D_1} (Q_x - P_y)\mathrm{d}x\mathrm{d}y, \tag{30}$$
where $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$.

On the other hand, due to Green's formula, we have

$$\int\int_D (Q_x - P_y)\mathrm{d}x\mathrm{d}y = \oint_k (P\mathrm{d}x + Q\mathrm{d}y).$$

Bearing in mind the above formula we obtain the statements that follow.

Theorem 6. *The relation between the circulation* (26) *and the flux* (30) *is determined by*

$$T = -2\cos\varphi C.$$

Corollary 1. *The relation between the circulation C and the flux T is*

(a) $T = -C$, *in case* $\varphi = \frac{\pi}{3}$;

(b) $T = C$, *in case* $\varphi = \frac{2\pi}{3}$.

4. Circles in the Two-Plane β_2

Lemma 2 ([3])**.** *Let* β_2 *be the 2-plane spanned by unit vectors u and Su. The system of vectors* $\{e_1, e_2\}$*, determined by the equalities*

$$e_1 = \frac{1}{\sqrt{2(1+\cos\varphi)}}(u + Su), \quad e_2 = \frac{1}{\sqrt{2(1-\cos\varphi)}}(-u + Su), \tag{31}$$

is an orthonormal basis of β_2 *with respect to g.*

The coordinate system p_{xy} on β_2, such that e_1 is on the axis p_x and e_2 is on the axis p_y, is orthonormal. Due to (10), we ascertain that the system $\{e_1, e_2\}$ satisfies the following equalities:

$$\tilde{g}(e_1, e_1) = \frac{2\cos\varphi + 1}{1+\cos\varphi}, \quad \tilde{g}(e_2, e_2) = \frac{2\cos\varphi - 1}{1-\cos\varphi}, \quad \tilde{g}(e_1, e_2) = 0. \tag{32}$$

A circle k_2 in β_2 centered at the origin p, with respect to $\tilde{g}$ on (M, g, S), is defined by (7). The equation of k_2 with respect to p_{xy} is obtained as below.

Theorem 7 ([3])**.** *Let* $\tilde{g}$ *be the associated metric on* (M, g, S) *and let* $\beta_2 = \{u, Su\}$ *be a 2-plane in* T_pM *with an orthonormal basis* (31)*. If* p_{xy} *is a coordinate system, such that* $e_1 \in p_x$, $e_2 \in p_y$*, then the equation of the circle* (7) *in* β_2 *is given by*

$$\frac{2\cos\varphi + 1}{1+\cos\varphi}x^2 + \frac{2\cos\varphi - 1}{1-\cos\varphi}y^2 = R^2. \tag{33}$$

The only closed curve k_2, determined by (33), is an ellipse in terms of g with parameters:

Case (A) $\varphi \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ and $R^2 > 0$;

Case (B) $\varphi \in \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$ and $R^2 < 0$.

The two-plane β_2 is constructed on space-like vectors in Case (A), and β_2 is constructed on time-like vectors in Case (B).

4.1. Length of a Circle with Respect to $\tilde{g}$

Firstly, we consider Case (A). The circle (7) has a radius $R > 0$ and φ satisfies

$$\frac{\pi}{4} < \varphi < \frac{\pi}{3}. \tag{34}$$

Theorem 8. *The circle* k_2 *with* (34) *and a radius* $R > 0$ *has length*

$$L = 2\pi R. \tag{35}$$

Proof. The radius vector v of an arbitrary point on the curve k_2 is $v = xe_1 + ye_2$. Then, $\mathrm{d}v = \mathrm{d}xe_1 + \mathrm{d}ye_2$ is a tangent vector on k_2. The length L of k_2, with respect to $\tilde{g}$, is

$$\mathrm{d}L = \sqrt{\tilde{g}(\mathrm{d}v, \mathrm{d}v)}.$$

From (32), we find

$$\tilde{g}(\mathrm{d}v, \mathrm{d}v) = \sqrt{\frac{2\cos\varphi + 1}{1 + \cos\varphi}\mathrm{d}x^2 + \frac{2\cos\varphi - 1}{1 - \cos\varphi}\mathrm{d}y^2}.$$

Then, we obtain

$$L = \oint_{k_2} \sqrt{\frac{2\cos\varphi + 1}{1 + \cos\varphi}\mathrm{d}x^2 + \frac{2\cos\varphi - 1}{1 - \cos\varphi}\mathrm{d}y^2}. \tag{36}$$

We substitute

$$x = R\sqrt{\frac{1 + \cos\varphi}{2\cos\varphi + 1}}\cos t, \ y = R\sqrt{\frac{1 - \cos\varphi}{2\cos\varphi - 1}}\sin t, \ t \in [0, 2\pi]$$

into (36) and obtain

$$L = \int_0^{2\pi} \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} dt,$$

which implies (35). □

Now, we consider Case (B). The circle k_2 has a radius $R = ri, \ r > 0, \ i^2 = -1$ and the angle φ satisfies

$$\frac{2\pi}{3} < \varphi < \frac{3\pi}{4}. \tag{37}$$

The Equation (33) transforms into

$$\frac{2\cos\varphi + 1}{1 + \cos\varphi}x^2 + \frac{2\cos\varphi - 1}{1 - \cos\varphi}y^2 = -r^2. \tag{38}$$

By calculations similar to those of Case (A), we find the integral (36) over k_2 with (38). Using the substitutions

$$x = \sqrt{-\frac{1 + \cos\varphi}{2\cos\varphi + 1}}r\cos t, \ y = \sqrt{-\frac{1 - \cos\varphi}{2\cos\varphi - 1}}r\sin t, \ t \in [0, 2\pi],$$

we calculate the length of k_2 and formulate the next proposition.

Proposition 3. *The circle k_2 with (37) and a radius $R = ri, \ r > 0, \ i^2 = -1$ has an imaginary length*

$$L = 2\pi R.$$

4.2. Area of a Circle with Respect to $\tilde{g}$

For Case (A) we state the following.

Theorem 9. *The area A'' of the circle k_2 with (34) and a radius $R > 0$ is*

$$A'' = \pi R^2. \tag{39}$$

Proof. Let us denote, by $\widetilde{\cos}\theta$, the cosine of $\theta = \angle(e_1, e_2)$ with respect to $\tilde{g}$, which is

$$\widetilde{\cos}\theta = \frac{\tilde{g}(e_1, e_2)}{\sqrt{\tilde{g}(e_1, e_1)}\sqrt{\tilde{g}(e_2, e_2)}}.$$

Then, using (32), we derive $\widetilde{\cos}\theta = 0$, which implies

$$\widetilde{\sin}\theta = 1. \tag{40}$$

In the coordinate plane p_{xy}, we construct a parallelogram with locus vectors $\mathrm{d}xe_1$ and $\mathrm{d}ye_2$. For its area A'' with respect to $\tilde{g}$ we obtain

$$\mathrm{d}A'' = \sqrt{\tilde{g}(\mathrm{d}xe_1, \mathrm{d}xe_1)}\sqrt{\tilde{g}(\mathrm{d}ye_2, \mathrm{d}ye_2)}\widetilde{\sin}\theta.$$

We apply (32) and (40) in the above equality and obtain

$$\mathrm{d}A'' = \frac{\sqrt{4\cos^2\varphi - 1}}{\sin\varphi}\mathrm{d}x\mathrm{d}y. \tag{41}$$

We integrate (41) over the region D_2 inside k_2 and calculate

$$A'' = \frac{\sqrt{4\cos^2\varphi - 1}}{\sin\varphi}\int\int_{D_2}\mathrm{d}x\mathrm{d}y, \tag{42}$$

with

$$D_2 : \frac{2\cos\varphi + 1}{1 + \cos\varphi}x^2 + \frac{2\cos\varphi - 1}{1 - \cos\varphi}y^2 \leq R^2.$$

We substitute

$$x = \sqrt{\frac{1 + \cos\varphi}{2\cos\varphi + 1}}R\rho\cos t,\ y = \sqrt{\frac{1 - \cos\varphi}{2\cos\varphi - 1}}R\rho\sin t,\ t \in [0, 2\pi],\ \rho \in [0, 1],$$

and Jacobian $\triangle = \dfrac{\sin\varphi}{\sqrt{4\cos^2\varphi - 1}}R^2\rho$ into (42). Finally we get (39). □

Now, we consider Case (B).

Proposition 4. *The area A'' of the circle k_2 with (34) and a radius $R = ri,\ r > 0,\ i^2 = -1$ has a negative value*

$$A'' = \pi R^2. \tag{43}$$

Proof. The circle k_2 has Equation (38), with conditions (37) and a radius $R = ri$, where $r > 0$, $i^2 = -1$. By calculations analogous to the previous case, we find that the area of k_2 is given by

$$A'' = \frac{\sqrt{4\cos^2\varphi - 1}}{\sin\varphi}\int\int_{D_2}\mathrm{d}x\mathrm{d}y, \tag{44}$$

with

$$D_2 : -\frac{2\cos\varphi + 1}{1 + \cos\varphi}x^2 - \frac{2\cos\varphi - 1}{1 - \cos\varphi}y^2 \leq r^2.$$

We substitute

$$x = \sqrt{-\frac{1 + \cos\varphi}{2\cos\varphi + 1}}r\rho\cos t,\ y = \sqrt{-\frac{1 - \cos\varphi}{2\cos\varphi - 1}}r\rho\sin t,\ t \in [0, 2\pi],\ \rho \in [0, 1],$$

and Jacobian $\triangle = \dfrac{\sin\varphi}{\sqrt{4\cos^2\varphi - 1}} r^2\rho$ into the integral (44) and obtain $A'' = -\pi r^2$, which implies (43). □

4.3. Circulation and Flux with Respect to $\tilde{g}$

We consider a closed curve k_2 in β_2, given by

$$x = x(t), \quad y = y(t), \quad t \in [t_1, t_2], \tag{45}$$

where $x(t_1) = x(t_2), y(t_1) = y(t_2)$.

Let

$$F(x, y) = P(x, y)e_1 + Q(x, y)e_2 \tag{46}$$

be a vector force field on the curve k_2.

We denote, by D_2, the region inside k_2. For both cases (A) and (B) of ellipse (33) the following statements hold.

Theorem 10. *The circulation C of the force (46) along the curve (45) is expressed by*

$$C = \oint_{k_2} \left(\frac{2\cos\varphi + 1}{1 + \cos\varphi} P(x, y)\mathrm{d}x + \frac{2\cos\varphi - 1}{1 - \cos\varphi} Q(x, y)\mathrm{d}y \right), \tag{47}$$

where $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$.

Proof. For the circulation C of a vector force field F acting along the curve (45) we use (25), where $v = xe_1 + ye_2$ is the radius vector of a point on k_2. Therefore, we have

$$C = \oint_{k_2} \tilde{g}(F, \mathrm{d}v), \tag{48}$$

with a tangent vector $\mathrm{d}v = \mathrm{d}xe_1 + \mathrm{d}ye_2$ on k_2. Then, by virtue of (32) and (46), we obtain

$$\tilde{g}(F, \mathrm{d}v) = \left(\frac{2\cos\varphi + 1}{1 + \cos\varphi} P(x, y)\mathrm{d}x + \frac{2\cos\varphi - 1}{1 - \cos\varphi} Q(x, y)\mathrm{d}y \right). \tag{49}$$

Hence, (45), (48) and (49) imply (47). □

Theorem 11. *The flux T of the vector field* curl *F across the region D_2 inside the curve (45) is expressed by*

$$T = -2\cot^3\varphi\sqrt{4\cos^2\varphi - 1} \int\int_{D_2} (Q_x - P_y)\mathrm{d}x\mathrm{d}y, \tag{50}$$

where $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$.

Proof. We determine a vector w in T_pM by the equality

$$w = \frac{1}{\sin\varphi\sqrt{1 - 2\cos^2\varphi}} \left(\cos^2\varphi u - (\cos\varphi)Su + \sin^2\varphi S^2 u \right), \tag{51}$$

where $\varphi \in \left(\frac{\pi}{3}, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$. Then, using (1), (3), (9) and (31), we verify that

$$g(w, e_1) = g(w, e_2) = 0, \; g(w, w) = 1.$$

The coordinate system $Oxyz$, such that $e_1 \in Ox$, $e_2 \in Oy$, $w \in Oz$, is orthonormal.

We obtain the curl of F, determined by (46), by the equality $\mathrm{curl}\, F = (Q_x - P_y)w$. For the flux T of the vector field curl F across the region D_2 inside the curve (45) we have

$$T = \int\int_{D_2} \tilde{g}(\mathrm{curl}\, F, w)\mathrm{d}A''. \tag{52}$$

Now, with the help of (32) and (51), we calculate

$$\tilde{g}(w,w) = -\frac{2\cos^3\varphi}{\sin^2\varphi}.$$

Then, from (41) and (52), it follows (50). □

We introduce the following notations:

$$c_1 = \frac{2\cos\varphi + 1}{1+\cos\varphi}\oint_{k_2} Pdx, \quad c_2 = \frac{2\cos\varphi - 1}{1-\cos\varphi}\oint_{k_2} Qdy. \tag{53}$$

On the other hand, due to Green's formula, we have

$$\int\int_D P_y \mathrm{d}x\mathrm{d}y = -\oint_k P\mathrm{d}x, \quad \int\int_D Q_x \mathrm{d}x\mathrm{d}y = \oint_k Q\mathrm{d}y.$$

Bearing in mind the latter equalities we derive the next statement.

Theorem 12. *The relation between the circulation (47) and the flux (50) is determined by*

$$T = -2\cot^3\varphi\Big((1+\cos\varphi)\sqrt{\frac{2\cos\varphi - 1}{2\cos\varphi + 1}}c_1 + (1-\cos\varphi)\sqrt{\frac{2\cos\varphi + 1}{2\cos\varphi - 1}}c_2\Big),$$

where c_1 and c_2 are given in (53).

5. Conclusions

In this paper, we investigated the properties of special two-planes β_1 and β_2 of T_pM of a four-dimensional Riemannian manifold (M, g, S), equipped with an additional indefinite metric $\tilde{g}(\cdot,\cdot) = g(\cdot, S\cdot) + g(S\cdot,\cdot)$. In these two-planes, circles, with respect to $\tilde{g}$, are transformed into closed curves in terms of g. Therefore, we can consider analogs of well-known formulae, such as circulation of a vector force field along the curve and flux of the curl of a vector force field across the curve. It turns out that the length and area, calculated with respect to the indefinite metric, of the circles in β_1 and β_2 are the same as in the Euclidean space.

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References

1. Dokuzova, I. Curvature properties of 4-dimensional Riemannian manifolds with a circulant structure. *J. Geom.* **2017**, *108*, 517–527. [CrossRef]
2. Dokuzova, I.; Razpopov, D. Four-dimensional almost Einstein manifolds with skew-circulant structures. *J. Geom.* **2020**, *111*, 9. [CrossRef]
3. Dzhelepov, G.; Dokuzova, I.; Razpopov, D. Spheres and circles with respect to an indefinite metric of a 4-dimensional Riemannian manifold with skew-circulant structures. *arXiv* **2023**, arXiv:2301.03675.

4. Davis, P.J. *Circulant Matrices*; A Wiley-Interscience Publication. Pure and Applied Mathematics; John Wiley and Sons: New York, NY, USA, 1979.
5. Garayar-Leyva, G.G.; Osman, H.; Estrada-López, J.J.; Moreira-Tamayo, O. Skew-Circulant-Matrix-Based Harmonic-Canceling Synthesizer for BIST Applications. *Sensors* **2022**, *22*, 2884. [CrossRef] [PubMed]
6. Gray, R.M. Toeplitz and circulant matrices: A Review, Found. *Trends Commun. Inf. Theory* **2006**, *2*, 155–239. [CrossRef]
7. Liu, Z.; Chen, S.; Xu, W.; Zhang, Y. The eigen-structures of real (skew) circulant matrices with some applications. *Comp. Appl. Math.* **2019**, *38*, 178. [CrossRef]
8. Ng, M.K. Circulant and skew-circulant splitting methods for Toeplitz systems. *J. Computat. Appl. Math.* **2003**, *159*, 101–108. [CrossRef]
9. Zhaolin, J.; Yao, J.; Lu, F. On Skew Circulant Type Matrices Involving Any Continuous Fibonacci Numbers. *Abst. Appl. Anal.* **2014**, *2014*, 483021.
10. Li, Y.; Abolarinwa, A.; Alkhaldi, A.; Ali, A. Some Inequalities of Hardy Type Related to Witten-Laplace Operator on Smooth Metric Measure Spaces. *Mathematics* **2022**, *10*, 4580. [CrossRef]
11. Li, Y.; Ganguly, D. Kenmotsu Metric as Conformal η-Ricci Soliton. *Mediterr. J. Math.* **2023**, *20*, 193. [CrossRef]
12. Li, Y.; Srivastava, S.K.; Mofarreh, F.; Kumar, A.; Ali, A. Ricci Soliton of CR-Warped Product Manifolds and Their Classifications. *Symmetry* **2023**, *15*, 976. [CrossRef]
13. Boothby, W.M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*, 2nd ed.; Pure and Applied Mathematics, 120; Academic Press, Inc.: Orlando, FL, USA, 1986.
14. Clark, P. Green's Theorem. Available online: http://alpha.math.uga.edu/~pete/handouteight.pdf (accessed on 4 March 2023).
15. do Carmo, M.P. *Differential Forms and Applications. Integration on Manifolds*; Universitext; Springer: Berlin/Heidelberg, Germany, 2012.
16. Gupta, V.G.; Sharma, P. Differential forms and its application. *Int. J. Math. Anal. (Ruse)* **2008**, *2*, 1051–1060.
17. Parkinson, C. The Elegance of Differential Forms in Vector Calculus and Electromagnetics. Master's Thesis, University of Chester, Chester, UK, 2014.
18. Petrello, R. Stokes' Theorem. Master's Thesis, California State University, Long Beach, CA, USA, 1998.
19. Abe, N.; Nakanishi, Y.; Yamaguchi, S. Circles and spheres in pseudo-Riemannian geometry. *Aequ. Math.* **1990**, *39*, 134–145. [CrossRef]
20. Holmes, R.D.; Thompson, A. N-dimensional area and content in Minkowski spaces. *Pac. J. Math.* **1979**, *85*, 77–110. [CrossRef]
21. Ikawa, T. On curves and submanifolds in an indefinite-Riemannian manifold. *Tsukuba J. Math.* **1985**, *9*, 353–371. [CrossRef]
22. Lopez, R. Differential geometry of curves and surfaces in Lorentz-Minkowski space. *Int. Electron. J. Geom.* **2014**, *7*, 44–107. [CrossRef]
23. Mustafaev, Z. The ratio of the length of the unit circle to the area of the unit disk in Minkowski planes. *Proc. Am. Math. Soc.* **2005**, *133*, 1231–1237. [CrossRef]