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THREE-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH CIRCULANT STRUCTURES

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ABSTRACT. We consider a 3-dimensional Riemannian manifold M with two circulant structures – a metric g and an additional structure q with $q^3 = \mathrm{id}$. The structure q is compatible with g such that an isometry is induced in any tangent space of M. We obtain some curvature properties of this manifold (M, g, q) and give an example of such a manifold.

1. INTRODUCTION

Circulant matrices occur in many areas of the applied mathematics. For instance, they are particulary useful in the Vibration analysis, Linear codes, Geometry, Graph theory, etc. (see [3], [4], [6], [8]). This motivates us to equip differentiable manifolds with additional structures which are represented by circulant matrices.

In differential geometry, essential results are associated with the sectional curvatures of some characteristic 2-planes of the tangent space of the manifolds with additional structures, (for example [2], [5], [7]). Another important problem is the obtaining of explicit examples of the constructed manifolds.

The main aim of the present paper is to study the differential geometry of 3-dimensional Riemannian manifolds equipped with an endomorphism q whose third power is the identity. Moreover, the metric g and the structure q are represented by circulant matrices.

The paper is organized as follows. In Sect. 2, we consider a 3-dimensional Riemannian manifold M with a circulant metric g and a circulant structure q satisfying $q^3 = \mathrm{id}$, i.e. a manifold (M, g, q). Also we recall necessary facts about such manifolds and about a q-*basis* of the tangent space T_pM, $p \in M$. In Sect. 3, we calculate the components of the curvature tensor R with respect to the Levi-Civita connection of g. In Sect. 4, we consider two special properties of R with respect to q and the consequences for some sectional curvatures. In Sect. 5, we obtain an explicit example.

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2. Preliminaries

Let M be a 3-dimensional manifold with a Riemannian metric g. Let the components of the metric g at an arbitrary point $p(X^1, X^2, X^3) \in M$ form the following circulant matrix

$$(g_{ij}) = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, \tag{2.1}$$

where A and B are smooth functions of X^1, X^2, X^3.

We assume that

$$A > B > 0. \tag{2.2}$$

Then the conditions to be a positive definite metric g are satisfied:

$$A > 0, \quad \begin{vmatrix} A & B \\ B & A \end{vmatrix} = (A - B)(A + B) > 0,$$

$$\begin{vmatrix} A & B & B \\ B & A & B \\ B & B & A \end{vmatrix} = (A - B)^2(A + 2B) > 0.$$

Let q be an endomorphism in the tangent space T_pM, whose coordinate matrix with respect to a basis $\{e_i\}$ of T_pM is

$$(q_i^{\cdot j}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Then

$$q^3 = \mathrm{id}.$$

We denote by (M, g, q) the manifold M equipped with the metric g and the structure q, which are defined by (2.1) – (2.3).

Further, x, y, z, u will stand for arbitrary elements of the algebra on the smooth vector fields on M or vectors in the tangent space T_pM. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3\}$.

In [1] it is proved that the structure q of the manifold (M, g, q) is an isometry with respect to the metric g, i.e.

$$g(qx, qy) = g(x, y). \tag{2.4}$$

Definition 2.1. *A basis of type $\{x, qx, q^2x\}$ of T_pM is called a q-basis. In this case we say that the vector x induces a q-basis of T_pM. Similarly, a basis $\{x, qx\}$ of a 2-plane $\alpha = \{x, qx\}$ is called a q-basis.*

In [1] it is verified that

(i) A vector $x = (x^1, x^2, x^3)$ induces a q-basis in T_pM if and only if

$$3x^1x^2x^3 \neq (x^1)^3 + (x^2)^3 + (x^3)^3;$$

(ii) If a vector x induces a q-basis of T_pM and $\varphi = \angle(x, qx)$, then

$$\angle(x, qx) = \angle(qx, q^2x) = \angle(x, q^2x) = \varphi, \quad \varphi \in \big(0, \frac{2\pi}{3}\big);$$

(iii) An orthogonal q-basis of T_pM exists.

3. The components of the curvature tensor

The Levi-Civita connection on a Riemannian manifold is denoted by ∇. For the Christoffel symbols Γ_{ij}^{s} of ∇ it is well known that

$$2\Gamma_{ik}^{h}=g^{ht}(\partial_{i}g_{tk}+\partial_{k}g_{ti}-\partial_{t}g_{ik}), \tag{3.1}$$

where g^{ij} are the components of the inverse matrix of (g_{ij}).

The curvature tensor R of ∇ is defined by

$$R(x,y)z=\nabla_{x}\nabla_{y}z-\nabla_{y}\nabla_{x}z-\nabla_{[x,y]}z$$

and the local components of R are

$$R_{ijk}^{h}=\partial_{j}\Gamma_{ik}^{h}-\partial_{k}\Gamma_{ij}^{h}+\Gamma_{ik}^{t}\Gamma_{tj}^{h}-\Gamma_{ij}^{t}\Gamma_{tk}^{h}. \tag{3.2}$$

The corresponding tensor R of type $(0,4)$ is determined as follows

$$R(x,y,z,u)=g(R(x,y)z,u).$$

For (M,g,q), we denote $D=(A-B)(A+2B)$ and

$$A_{i}=\frac{\partial A}{\partial X^{i}}\ ,\quad B_{i}=\frac{\partial B}{\partial X^{i}}\ , \tag{3.3}$$

where A and B are the functions from (2.1).

The inverse matrix of g is

$$(g^{ij})=\frac{1}{D}\begin{pmatrix}A+B & -B & -B\\ -B & A+B & -B\\ -B & -B & A+B\end{pmatrix}. \tag{3.4}$$

Then by direct calculations, having in mind (2.1), (3.1), (3.2), (3.3) and (3.4), we obtain

Theorem 3.1. *The nonzero components of the curvature tensor R of type $(0,4)$ of the manifold (M,g,q) are*

$$\begin{aligned}R_{1212}&=\frac{1}{2}(2B_{21}-A_{11}-A_{22})\\&+\frac{A+B}{4D}\Big(2A_{3}B_{2}-A_{3}^{2}+(B_{1}-B_{2}-B_{3})(B_{1}+B_{2}-B_{3})\Big)\\&-\frac{2B}{4D}\Big((A_{1}-B_{2})(B_{1}+B_{2}-B_{3})-A_{1}A_{3}+A_{3}B_{2}\Big),\end{aligned}$$

$$\begin{aligned}R_{1313}&=\frac{1}{2}(2B_{31}-A_{11}-A_{33})\\&+\frac{A+B}{4D}\Big(2A_{2}B_{3}-A_{2}^{2}+(-B_{1}+B_{2}+B_{3})(-B_{1}+B_{2}-B_{3})\Big)\\&-\frac{2B}{4D}\Big((A_{1}-B_{3})(B_{1}-B_{2}+B_{3})-A_{1}A_{2}+A_{2}B_{3}\Big),\end{aligned}$$

$$\begin{aligned}R_{2323}&=\frac{1}{2}(2B_{23}-A_{22}-A_{33})\\&+\frac{A+B}{4D}\Big(2B_{3}A_{1}-A_{1}^{2}+(B_{1}-B_{2}+B_{3})(B_{1}-B_{2}-B_{3})\Big)\\&-\frac{2B}{4D}\Big((A_{2}-B_{3})(B_{2}-B_{1}+B_{3})-A_{1}A_{2}+A_{1}B_{3}\Big),\end{aligned}$$

$$\begin{aligned}
R_{1213} &= \frac{1}{2}(B_{21}+B_{31}-B_{11}-A_{23})\\
&+\frac{A+B}{4D}\Big(A_1(B_2-B_3+B_1)+2B_3(B_3-B_2-B_1)+A_2A_3)\Big)\\
&-\frac{B}{4D}\Big(A_1^2+A_2^2+A_3^2+2A_1(A_2-B_3)-2A_2B_3\\
&-2A_3(B_1-B_3)+(B_1-B_2-B_3)(B_1+B_2-B_3)\Big),
\end{aligned}$$

$$\begin{aligned}
R_{1223} &= \frac{1}{2}(B_{22}-B_{12}-B_{23}+A_{13})\\
&+\frac{A+B}{4D}\Big(A_2(B_2+B_3-B_1)-(2B_3-A_1)(2B_2-A_3)\Big)\\
&-\frac{B}{4D}\Big(A_2^2-A_1^2+A_3^2+2A_1(B_2+B_3)+2A_2(B_2-B_3)\\
&+2A_3(B_3-B_1)-4B_2B_3+(B_1+B_2-B_3)(B_1-B_2-B_3)\Big),
\end{aligned}$$

$$\begin{aligned}
R_{1323} &= \frac{1}{2}(B_{23}-B_{33}+B_{13}-A_{12})\\
&+\frac{A+B}{4D}\Big((2B_2-A_1)(2B_3-A_2)-A_3(-B_1+B_2+B_3)\Big)\\
&-\frac{B}{4D}\Big(A_1^2-A_2^2-A_3^2-2A_1(B_2+B_3)+2A_2(B_1-B_2)\\
&+2A_3(B_2-B_3)+4B_2B_3+(-B_1+B_2+B_3)(B_1-B_2+B_3)\Big).
\end{aligned}$$

The rest of the nonzero components are obtained by the properties

$$R_{ijkh}=R_{khij},\ R_{ijkh}=-R_{jikh}=-R_{ijhk}\ .$$

4. Some sectional curvatures

In [1], for (M,g,q) it is proved that $\nabla q=0$ implies

$$R(x,y,qz,qu)=R(x,y,z,u). \tag{4.1}$$

Therefore it follows the identity

$$R(qx,qy,qz,qu)=R(x,y,z,u), \tag{4.2}$$

which defines a more general class of manifolds (M,g,q) than the class with the condition $\nabla q=0$.

Let $\{x,y\}$ be a non-degenerate 2-plane spanned by vectors $x,y\in T_pM$, $p\in M$. Then its sectional curvature is

$$\mu(x,y)=\frac{R(x,y,x,y)}{g(x,x)g(y,y)-g^2(x,y)}. \tag{4.3}$$

Proposition 4.1. *Let (M,g,q) be a manifold with property* (4.2) *and a vector x induce a q-basis. Then*

$$\mu(x,qx)=\mu(qx,q^2x)=\mu(x,q^2x).$$

Proof. From (4.2) we get

$$R(q^2x, q^2y, q^2z, q^2u) = R(qx, qy, qz, qu) = R(x, y, z, u). \tag{4.4}$$

In (4.4) we substitute qx for y, x for z and qx for u and we find

$$R(q^2x, x, q^2x, x) = R(qx, q^2x, qx, q^2x) = R(x, qx, x, qx). \tag{4.5}$$

Then, from (2.4) and (4.3) it follows (4.3). □

Let x induce a q-basis of T_pM and $\sigma = \{x, qx\}$ be a 2-plane. It is easy to see that if $y \in \sigma$ and $y \neq x$, then $qy \notin \sigma$. Consequently, σ has only two q-bases: $\{x, qx\}$ and $\{-x, -qx\}$. That's why the sectional curvature $\mu(x, qx)$ depends only on $\varphi = \angle(x, qx)$. So, we denote $\mu(x, qx) = \mu(\varphi)$.

Theorem 4.1. *Let* (M, g, q) *be a manifold with property* (4.2). *If a vector* u *induces a* q*-basis, then*

$$\mu(\varphi) = \frac{1 - 2\cos\varphi}{1 + \cos\varphi}\mu\Big(\frac{\pi}{2}\Big) + \frac{3\cos\varphi}{1 + \cos\varphi}\mu\Big(\frac{\pi}{3}\Big)\ , \tag{4.6}$$

where $\varphi = \angle(u, qu)$.

Proof. In (4.4) we substitute qx for y, q^2x for z and x for u and we get

$$R(q^2x, x, qx, q^2x) = R(qx, q^2x, x, qx) = R(x, qx, q^2x, x). \tag{4.7}$$

Let a vector x induce an orthonormal q-basis. If $u = \alpha x + \beta qx + \gamma q^2x$, where $\alpha, \beta, \gamma \in \mathbb{R}$, then $qu = \gamma x + \alpha qx + \beta q^2x$. Due to the linear properties of the curvature tensor R, we obtain

$$\begin{aligned} R(u, qu, u, qu) &= (\alpha^2 - \beta\gamma)^2 R(x, qx, x, qx) \\ &+ (\gamma^2 - \alpha\beta)^2 R(x, q^2x, x, q^2x) \\ &+ (\beta^2 - \alpha\gamma)^2 R(qx, q^2x, qx, q^2x) \\ &+ 2(\alpha^2 - \beta\gamma)(\gamma^2 - \alpha\beta) R(x, qx, q^2x, x) \\ &+ 2(\gamma^2 - \alpha\beta)(\beta^2 - \alpha\gamma) R(q^2x, x, qx, q^2x) \\ &+ 2(\alpha^2 - \beta\gamma)(\beta^2 - \alpha\gamma) R(x, qx, qx, q^2x). \end{aligned}$$

Having in mind (4.5) and (4.7) we find

$$\begin{aligned} &R(u, qu, u, qu) = \Big((\alpha^2 - \beta\gamma)^2 \\ &+ (\gamma^2 - \alpha\beta)^2 + (\beta^2 - \alpha\gamma)^2\Big) R(x, qx, x, qx) \\ &+ 2\Big((\alpha^2 - \beta\gamma)(\gamma^2 - \alpha\beta) + (\gamma^2 - \alpha\beta)(\beta^2 - \alpha\gamma) \\ &+ (\alpha^2 - \beta\gamma)(\beta^2 - \alpha\gamma)\Big) R(x, qx, q^2x, x). \end{aligned} \tag{4.8}$$

Since $\{x, qx, q^2x\}$ is an orthonormal q-basis, we have

$$g(u, u) = g(qu, qu) = \alpha^2 + \beta^2 + \gamma^2, \quad g(u, qu) = \alpha\beta + \beta\gamma + \gamma\alpha.$$

We suppose that $g(u, u) = 1$. From (2.4) and (4.3) we get

$$\mu(\varphi) = \frac{R(u, qu, u, qu)}{1 - \cos^2\varphi}\ , \tag{4.9}$$

and

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \cos\varphi.$$

We express α, β, γ by $\cos\varphi$ as follows:

$$\begin{aligned}(\cos\varphi)^2 - \cos\varphi &= (\alpha^2 - \beta\gamma)(\gamma^2 - \alpha\beta) + (\gamma^2 - \alpha\beta)(\beta^2 - \alpha\gamma)\\ &+ (\alpha^2 - \beta\gamma)(\beta^2 - \alpha\gamma),\\ 1 - (\cos\varphi)^2 &= (\alpha^2 - \beta\gamma)^2 + (\gamma^2 - \alpha\beta)^2 + (\beta^2 - \alpha\gamma)^2.\end{aligned}$$

Then, from (4.8) and (4.9), we obtain

$$\mu(\varphi) = \mu\Big(\frac{\pi}{2}\Big) + \frac{2\cos\varphi}{1+\cos\varphi} R(x, qx, x, q^2x). \tag{4.10}$$

In (4.10) we substitute $\frac{\pi}{3}$ for φ and we find

$$R(x, qx, x, q^2x) = \frac{3}{2}\Big(\mu\Big(\frac{\pi}{3}\Big) - \mu\Big(\frac{\pi}{2}\Big)\Big).$$

The last result and (4.10) imply (4.6). □

Corollary 4.1. *Let* (M, g, q) *be a manifold with property* (4.1). *If a vector* u *induces a* q*-basis, then*

$$\mu(\varphi) = \frac{1-\cos\varphi}{1+\cos\varphi}\mu\Big(\frac{\pi}{2}\Big)\ , \tag{4.11}$$

where $\varphi = \angle(u, qu)$.

Proof. Since (4.1) is valid, we find

$$R(x, y, z, u) = R(x, y, qz, qu) = R(x, y, q^2z, q^2u).$$

In the latter equalities we substitute qx for y, x for z and qx for u and we get

$$R(x, qx, x, q^2x) = -R(x, qx, x, qx). \tag{4.12}$$

If we suppose that $\{x, qx, q^2x\}$ is an orthonormal q-basis, then from (4.10) and (4.12) it follows (4.11). □

5. An example of (M, g, q)

In [1] it is proved that, the structure q is parallel with respect to the Levi-Civita connection ∇ of g on a manifold (M, g, q) if and only if the gradients of A and B satisfy the following equality:

$$\mathrm{grad}A = \mathrm{grad}B \begin{pmatrix} -1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1 \end{pmatrix}. \tag{5.1}$$

In this section we discuss an example of a manifold (M, g, q) which satisfies (4.2), but doesn't satisfy (5.1).

Theorem 5.1. *The property* (4.2) *of the manifold* (M, g, q) *is equivalent to the conditions*

$$R_{1212} = R_{1313} = R_{2323}, \quad R_{1213} = R_{1323} = -R_{1223}, \tag{5.2}$$

where R_{ijkh} *are the local components of the curvature tensor* R *of type* $(0, 4)$.

Proof. The local form of (4.2) is

$$R_{tslm}q_i^t q_j^s q_k^l q_h^m = R_{ijkh}. \tag{5.3}$$

From (2.1) and (5.3) we find

$$\begin{aligned} &R_{1212} = R_{2323}, \quad R_{1313} = R_{2121}, \\ &R_{2321} = R_{1213}, \quad R_{2331} = R_{1223}, \\ &R_{2131} = R_{1323}, \quad R_{3131} = R_{2323}, \end{aligned} \tag{5.4}$$

which implies (5.2).

Vice versa, from (5.2) it follows (5.4). Having in mind (2.1) we get (5.3). □

Let (M, g, q) be a manifold with

$$A = 2X^1 \ , \ B = 2X^1 + X^2 + X^3, \tag{5.5}$$

where

$$2X^1 + X^2 + X^3 > 0, \quad X^2 + X^3 < 0.$$

Obviously, the condition (2.2) is satisfied. Due to (2.1), (3.4), (5.5) and Theorem 3.1 we obtain

$$\begin{aligned} &R_{1212} = R_{1313} = R_{2323} = -\frac{B}{(A-B)(A+2B)} \ , \\ &R_{1213} = R_{1323} = R_{1223} = 0. \end{aligned} \tag{5.6}$$

We check directly that the conditions (5.2) are valid, but the conditions (5.1) for the functions A and B are not valid.

Consequently, we obtain the following

Theorem 5.2. *The manifold (M, g, q) with (5.5) satisfies the curvature identity (4.2). Furthermore, the structure q is not parallel with respect to the Levi-Civita connection ∇ of g.*

The Ricci tensor ρ and the scalar curvature τ are given by the well-known formulas:

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \ \tau = g^{ij} \rho(e_i, e_j).$$

We obtain the components of ρ and the value of τ:

$$\begin{aligned} &\rho_{12} = \rho_{13} = \rho_{23} = \frac{B^2}{(A-B)^2(A+2B)^2} \ , \\ &\rho_{11} = \rho_{22} = \rho_{33} = \frac{2B(A+B)}{(A-B)^2(A+2B)^2} \ , \end{aligned} \tag{5.7}$$

$$\tau = \frac{6AB}{(A-B)^3(A+2B)^2} \ . \tag{5.8}$$

Therefore, we arrive at the following

Proposition 5.1. *For the manifold (M, g, q) with (5.5), the following assertions are valid:*

(i) *The components of the curvature tensor R are (5.6), i.e. M is not a flat manifold;*
(ii) *The components of the Ricci tensor ρ are (5.7);*
(iii) *The scalar curvature τ is (5.8).*

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